http://docenti.unisi.it/lucaregis
Financial Mathematics

Dr. Luca Regis • 2018/2019 • University of Siena

Last Revision: October 22, 2019

Table of Contents

Disclaimer

These lecture notes cover the program of the course of Financial Mathematics taught at the University of Siena, starting from a.y. 2016-17.
These notes are largely based on the contents of the Italian textbook by G. Castellani, M. De Felice and F. Moriconi, *Manuale di Finanza I. Tassi d'interesse*, il Mulino, Bologna, 2005 and in particular on chapters 1, 4-13.

1 Fundamentals of Financial Calculus

1.1 The financial contract

A *financial contract* is an agreement between two (or more) parties, whose object is the exchange of sums of money at predetermined dates. The contract specifies:

- the amounts (cash flows), explicitly or through a defined calculation rule and their unit of measure, i.e. the currency;
- the dates at which the exchanges occur and the unit of measure of time.

Usually, in the most common scheme, one party receives money at the beginning of the contract, and is obliged to return money back in the future. This party is defined as the *receiver, debtor*, or *issuer*. The other party, that delivers money to the counterpart at the start of the contract, in exchange for a future amount, is defined as the *lender, creditor*, or *investor*.

Bonds and mortgages are the most common financial contracts used to regulate exchanges in which one party receives financing from another party.

The fundamental element that characterizes all the financial contracts is the transfer of sums of money in time, that has its economic rationale in the fact that different agents have different consumption needs and profiles. The transfer of money in time is the element that determines the price of the transaction. This price is specified by the contract and arises practically because the lender will demand a compensation for giving up a part of his liquidity for some time. [1]

1.2 A two-date financial contract

Let us consider an example of such basic contract more in detail. At time $t = 0$ a debtor receives a sum $S > 0$, that he will be reimbursing at $t = 1$ together with an additional sum I, that prices the *time value of money* in the transaction.

I is the compensation received by the lender, and we call this the *interest*.

[1] Indeed, this is always the case if we assume that agents are profit maximizers (i.e. they always prefer more money to less) and that the cost of holding money is null.

We represent the exchange, from the point of view of the debtor, in the following way: $\{S, -(S+I)\}/\{0,1\}$, where the first couple represents the amounts involved in the exchange and the second couple represents the dates at which amounts will be exchanged. The same contract, from the point of view of the lender is represented as $\{-S, (S+I)\}/\{0,1\}$. Indeed, by signing the financial contract, the two parties agree on the intertemporal equivalence between the amount $W(0) = S$ at $t = 0$ and $W(1) = S + I$ at $t = 1$. Amounts that are received by the agent (inflows) have a positive sign, amounts that are paid (outflows) are negative. The following terminology is used:

- S is the *principal* or *invested capital*;
- I is the *interest*;
- $i = \frac{I}{S}$ is the *interest rate.*

It is natural, in our setting, since we are studying monetary exchanges, to assume that $I > 0$ (and $i > 0$ as a consequence) while this may not be the case under uncertainty or when real goods are exchanged.

Example 1.1. $S = 100, I = 2$, find i.

$$i = 2/100 = 0.02 = 2\%.$$

Example 1.2. $S = 100$, $i = 5\%$. Find I.

$$I = 100 * 5\% = 100 * 0.05 = 5.$$

1.3 A two-period financial contract

Let us now analyze a similar problem, involving a 2-period contract. The debtor, who receives the amount S at time $t = 0$, can now either return $S + I$ to the creditor at $t = 1$ or $S + I + I'$ at time $t = 2$.

Thus, the contract can take two forms for the debtor, either $\{S, -(S+I)\}/\{0,1\}$ or $\{S, -(S+I+I')\}/\{0,2\}$. Indeed, this is equivalent to the agreement, beyond

the $\{S, -(S+I)\}/\{0,1\}$ contract, to have the possibility to enter a transaction in which the debtor can further receive the amount $S+I$ at time $t=1$ with the obligation to return $S+I+I'$ at $t=2$. All the terms of the whole contract are agreed upon at $t=0$.
Hence, the operation $\{S, -(S+I+I')\}/\{0,2\}$ can be seen as the composition of two elementary transactions:

- an *immediate* (*spot*) transaction, $\{S, -(S+I)\}/\{0,1\}$, which occurs at time $t=0$, when the agreement is signed;
- a *deferred* (*forward*) transaction, $\{S+I, -(S+I+I')\}/\{1,2\}$, whose terms are also agreed upon before the first sum is exchanged.

Accordingly:

- the interest rate $i = \frac{I}{S}$ is a *spot interest rate*;
- the interest rate $i' = \frac{I'}{S+I}$ is a *forward interest rate*.

It is now important, in this context, to better specify the notion of a *value function* $W(t)$, which defines the value at time t of a sum initially valued $W(0)$. This function describes the financial equivalence between sums at different time instants, as defined by the rules specified in the financial contract. In this particular case, it is useful to describe, by means of the value function, the value of the sum S initially lent by the creditor:

- $W(0) = S$;
- $W(1) = S + I$;
- $W(2) = S + I + I'$.

Example 1.3. Let's consider an invested capital $S = 100000$€. The two parties have agreed on $i_1 = 2\%, i_2 = 3\%$. Find I_1 and I_2.

$$I_1 = 100000 * 2\% = 2000€$$

$$I_2 = i_2 * (S + I_1) = 3\% * (100000 + 2000) = 3060€.$$

Thus, the two operations, from the point of view of the lender, are described by $\{-100000, 102000\}/\{0,1\}$ and $\{-100000, 105060\}/\{0,2\}$ and by $\{100000, -102000\}$ $/\{0,1\}$ and $\{100000, -105060\}/\{0,2\}$ from the point of view of the debtor.

1.4 Simple interest

Let us consider a multi-period extension of the two-period contract we defined in the previous section. Suppose the contract rules specify that the interest I relative to each period is constant. In practice, the debtor can reimburse her creditor at any future (integer, for the moment) instant $n = 1, 2, ..., N$, adding to the initial capital S a constant fraction of the initial capital $i_S \times S$ for each period in which the debt has not been reimbursed. I, or equivalently i_S, uniquely defines the intertemporal equivalence of sums under this contract. Suppose the initial sum exchanged is $W(0) = S > 0$. Then:

- $W(0) = S$;
- $W(1) = S + i_S S = S + I$;
- $W(2) = S + 2i_S S = S + 2I$;
- $....W(n) = S + ni_S S = S + nI,...$;
- $...W(N) = S + Ni_S S = S + NI$.

The $N + 1$ terms that the value function defines are in arithmetic progression, with common difference $I = i_S S$: $W(n) = W(n-1) + I, t = 1, ...N$ and first term $W(0) = S$.
Hence, after t years, the value of the initial debt S is

$$W(n) = S(1 + i_S n), n = 0, 1...N. \tag{1.1}$$

The value function linearly increases with time. The equivalence law we have just defined is referred to as the *simple interest law*.

The interest rate i_n, relative to the yearly transaction starting at time $n-1$, is given by:

$$i_n = \frac{I}{S + (n-1)I} = \frac{W(n) - W(n-1)}{W(n-1)}.$$

Under the usual assumption that $I > 0$, the interest rate i_n is hyperbolically decreasing with time n. Notice that the interest rate i appearing in equation (1.1) is $i_1 = \frac{I}{S} = i_s$, i.e. the first-period (spot) interest rate. If the length of the period is one year, this rate is referred to as the *annual simple interest rate*.

Example 1.4. Let us consider an invested capital $S = 100$€, $N = 5$ and a simple interest law where $i = 3\%$. We have:

$$I = 100 * 3\% = 3;$$

$$W(0) = 100; W(1) = 100 + 3; W(2) = 100 + 2 * 3 = 100 + 3 + 3 = 103;$$

$$W(3) = 100 + 3 * 3 = 109; W(4) = 100 + 4 * 3 = 112; W(5) = 100 + 5 * 3 = 115.$$

The spot rate is equal to the annual simple interest rate: $i_1 = i = 3\%$. The forward rates are:

$$i_2 = \frac{3}{103} = 2.91\%; i_3 = \frac{3}{106} = 2.83\%;$$
$$i_4 = \frac{3}{109} = 2.75\%; i_5 = \frac{3}{112} = 2.68\%.$$

1.5 Compound interest

Suppose now that the N elementary one-period transactions involved in the contract described in the previous section have instead the same constant periodic interest rate $i_n = i_c$:

$$i_n = \frac{I(n)}{W(n-1)} = \frac{W(n) - W(n-1)}{W(n-1)} = i_c, \forall t = 1, ...N.$$

If the length of the period is one year, we call i_c the *annual compound interest rate*.

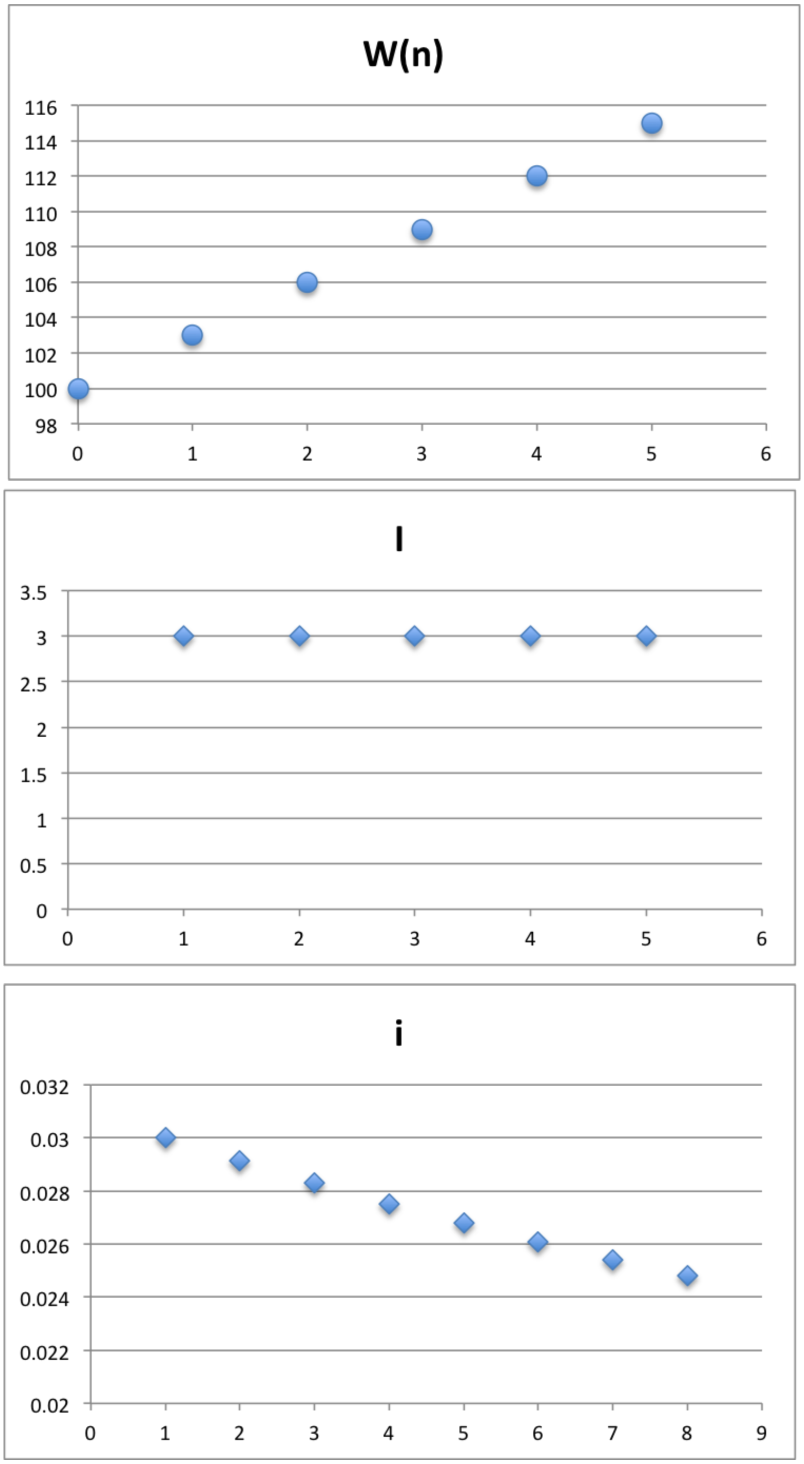

Figure 1.1: $W(n), I_n$ and i_n for the simple interest rate rule.

The sum S, after n years, from the equation above, has value

$$W(n) = i_c W(n-1) + W(n-1) = W(n-1)(1+i_c).$$

The value function is, if $i_c > 0$, exponentially increasing in time. Indeed, the $N+1$ terms of the value function form a geometric progression, where the common ratio of this sequence is $1+i_c$, as $\frac{W(n)}{W(n-1)} = 1+i_c$. Iterating the relation, a sum S at 0 has value at n equal to $W(n) = S(1+i_c)^n$.
The intertemporal equivalence law underlying the contract we described above is referred to as the *compound interest law*.
Let us now focus better on the value function, for an initial sum $W(0) = S$. Its values are

- $W(0) = S$;
- $W(1) = S + I(1) = S + i_c W(t-1) = S(1+i_c)$;
- $W(2) = W(1) + I(2) = S(1+i_c) + i_c(S(1+i_c)) = S(1+i_c)^2$;
- $...W(n) = W(n-1) + I(n) = S(1+i_c)^{n-1} + i_c S(1+i_c)^{n-1} = S(1+i_c)^n...$;
- $...W(N) = W(N-1) + I(N) = S(1+i_c)^{N-1} + i_c S(1+i_c)^{N-1} = S(1+i_c)^N$.

Example 1.5. Let us consider an invested capital $S = 100$, with $N = 5$ and $i = 3\%$. We have:

$$W(0) = 100; W(1) = 100(1+0.03) = 103; W(2) = 100(1+0.03)^2 = 106.09;$$

$$W(3) = 100(1+0.03)^3 = 109.27; W(4) = 100 * (1+0.03)^4 = 112.55;$$

$$W(5) = 100 * (1+0.03)^5 = 115.93.$$

In contrast with the simple interest case, where the periodic interest is constant, the periodic interest is computed as a fraction i_c of the accumulated value of the sum S at the beginning of each period, i.e. at time $n-1$. As it can be easily noticed, the periodic interest increases exponentially in time, provided $i_c > 0$:

$$I(1) = i_c S;$$

$$I(2) = W(2) - W(1) = i_c S(1 + i_c) = I(1)(1 + i_c);$$

$$I(n) = W(n) - W(n-1) = S(1+i_c)^n - S(1+i_c)^{n-1} = i_c S(1+i_c)^{n-1} = I(1)(1+i_c)^{n-1}.$$

What is the difference between a compound interest and a simple interest? Let us fix $i_s = i_c = i$ and compare the interests in the first two years, where the subscripts s and c denote the interests obtained with the simple and compound interest regime, respectively:

- $I_s(1) = iS$, $I_c(1) = iS$;
- $I_s(2) = iS$, $I_c(2) = iS(1 + i) = I(1)i + iS$;

The difference between $I_s(2)$ and $I_c(2)$ is $iI(1) = iI_s(1) = iI_c(1)$, that is the interest computed on the interest matured until the beginning of the period. The term "compound" refers precisely to the fact that the interest, at the end of each year, is compounded, i.e. it adds to the initial capital, contributing to the computation of the interests in the years to follow.

Example 1.6. Let us consider the simple interest case, and $i_s = 1\%, S = 100, I(n) = 100 * 1\% = 1, \forall n$:

$$W(0) = S = 100; W(1) = W(0) + I = 100 + 1 = 101.$$

$$W(2) = W(1) + I(2) = 101 + 1 = 102.$$

Let us now consider a compound interest $i_c = 1\%$, $S = 100$.

$$W(0) = S = 100; W(1) = S(1 + i_c) = 100(1.01) = 100 + 1 = 101.$$

$$W(2) = W(1) + I(2) = S(1 + i_c) + i_c S + i_c * i_c S = 101 + 1 + 1 * 1\% = 102.01.$$

The use of the simple vs. compound interest rate law depends on the agreement that rules the contract. In Italy, legal interests are computed based on – where not otherwise agreed or stated – a simple interest defined by law. Bank-activities related interests are usually simple when the length of the investment does not exceed the year, compound above the year.

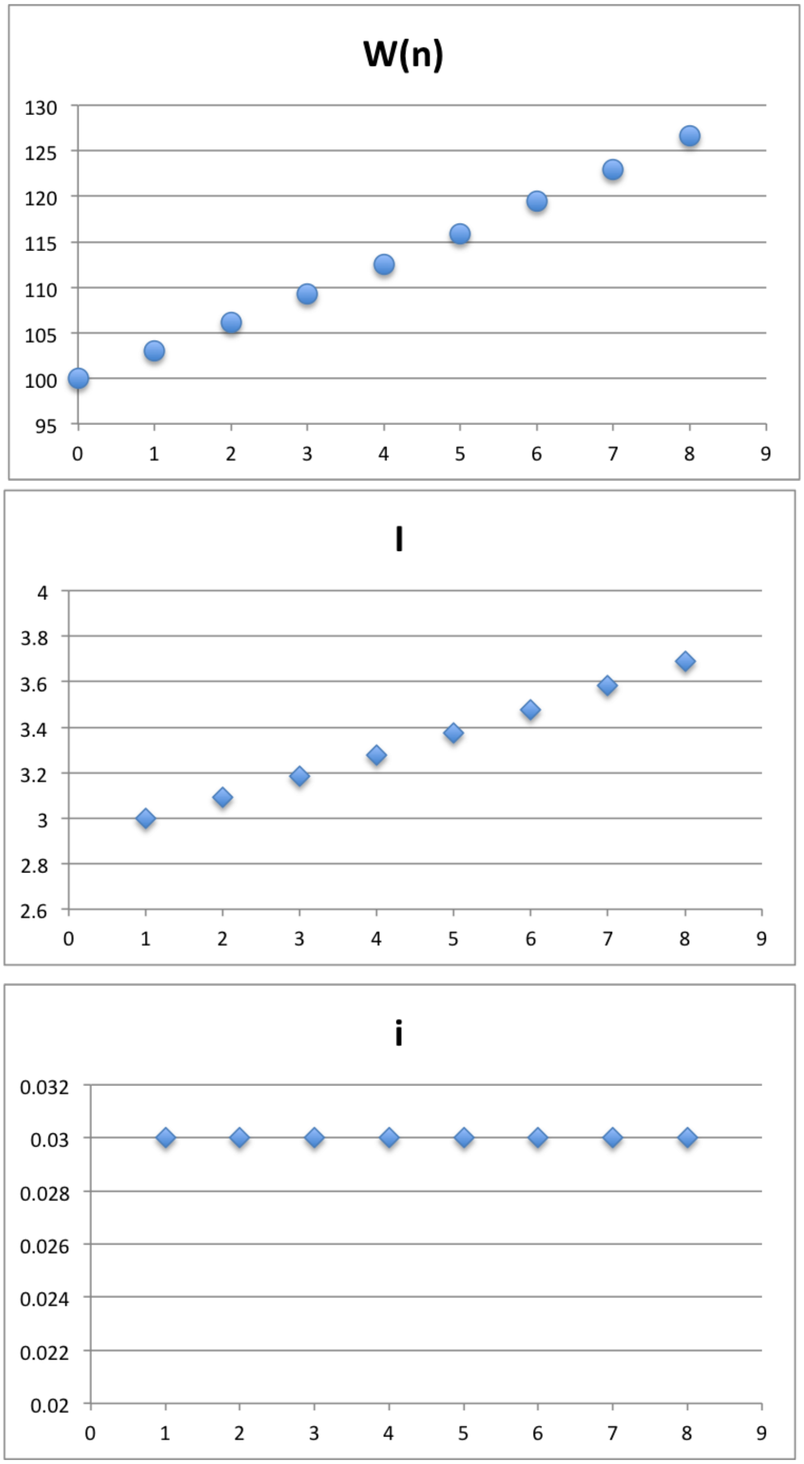

Figure 1.2: $W(n), I_n$ and i_n for the compound interest rate rule.

1.6 Linear law and exponential law

It is crucial now to extend the value functions we described above, defined so far on the set of natural numbers, to the whole positive real axis. This allows us to describe contracts whose payments may occur at any instant $t \geq 0$ (think for instance, of a bank account). That is, we will from now on shift the attention to a continuous-time description of the financial laws of intertemporal equivalence we defined in the previous sections. In practice, we define the two following financial laws:

1. *linear law*: $W_s(t) = S(1 + i_s t), t \geq 0$;

2. *exponential law*: $W_c(t) = S(1 + i_c)^t, t \geq 0$.

The function $m_s(i_s, t) = (1 + i_s t)$ is the *simple interest accumulation factor*, while the function $m_c(i_c, t) = (1+i_c)^t$ is referred to as the *compound interest accumulation factor*.
When $i_c = i_s = i > 0$, these two functions, or equivalently, W_s and W_c, assuming S equal for the two formulations, have the same value when $t = 0$ and $t = 1$. Moreover, being the exponential law convex, when $t < 1$, between 0 and 1 the linear law is always a majorant of the exponential law. This happens as well when $i < 0$.
More in general, when $i_c > 0$, $i_s > 0$ and $i_c \neq i_s$, there are two possible scenarios in the comparison between f_s and f_c. The exponential law can be either majorant of the linear law $\forall t > 0$ or there can be a point t^* such that it is majorant $\forall t > t^*$, while it is minorant when $t < t^*$. The emergence of one of the two scenarios depends on whether $m'_c(0) > m'_s(0)$, where m' denotes the derivative of f w.r.t. t. Being the linear law linear in i, $m'_s(0) = i_s = m'_s(t) \forall t \geq 0$. For the exponential law, $m'_c(t) = ln(1+i_c)(1+i_c)^t$. Hence, $m'_c(0) = ln(1+i_c)$. Summarizing, the exponential law is always majorant, whenever $ln(1 + i_c) > i_s$.

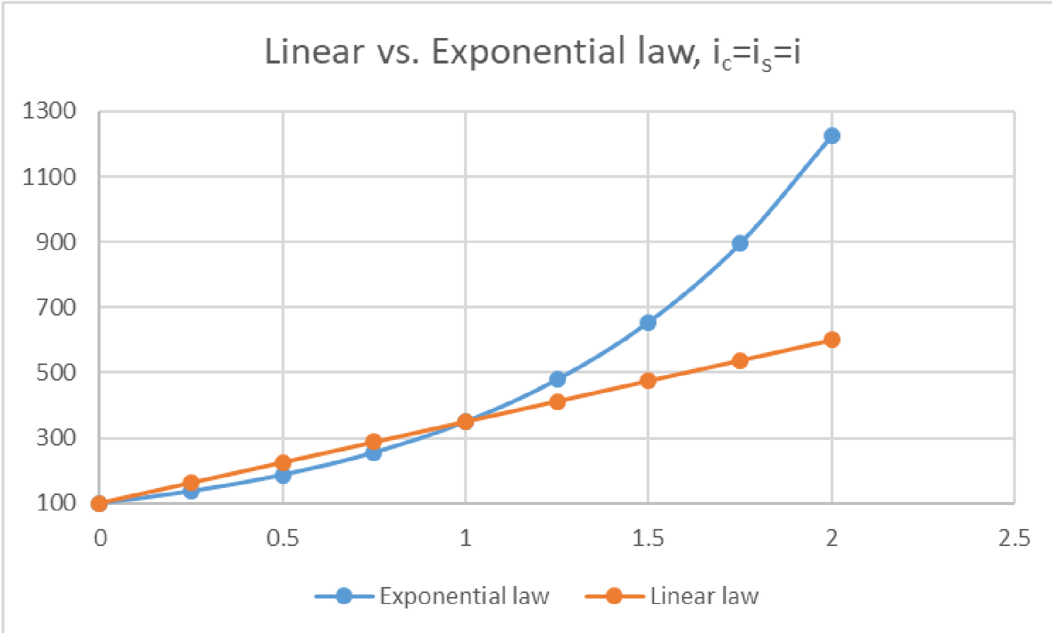

Figure 1.3: Exponential vs. linear law value function when $i_c = i_s = i$.

1.7 Fundamental quantities of financial mathematics

Let us now focus on the fundamental notions of financial mathematics. Consider a positive real-valued function $W(t)$, defined on the positive axis, i.e. for $t \geq 0$. As in the previous sections, this function represents the intertemporal equivalence law between money sums at different dates. Let us consider two time instants, $t \geq 0$ and $t' = t + \Delta t$, $\Delta t > 0$. Then:

- $\Delta W(t) = W(t + \Delta t) - W(t)$ is the *interest* in period $[t, t + \Delta t]$ of length Δt.
- $j(t, t + \Delta t) = \frac{\Delta W(t)}{W(t)}$ is the *rate of return* or *interest rate* in period $[t, t + \Delta t]$.

Notice that, while $\Delta W(t)$ is a quantity expressed in monetary terms, $j(t, t + \Delta t)$ is a number, usually expressed in percentage terms. Similarly to $j(t, t + \Delta t)$, we can define

- $d(t, t + \Delta t) = \frac{\Delta W(t)}{W(t+\Delta t)}$, the *anticipated rate of return* or interest rate in period $[t, t + \Delta t]$.

Then, we define

- $m(t, t + \Delta t) = \frac{W(t+\Delta t)}{W(t)}$, the *accumulation or capitalization factor*;

- $v(t, t+\Delta t) = \frac{W(t)}{W(t+\Delta t)}$, the *discount factor.*

These two quantities are the ratio of two monetary amounts, and have thus no dimension. If $W(t)$ is increasing in time, $m(t, t+\Delta t) > 1$ and $v(t, t+\Delta t) < 1$. $m(t, t+\Delta t)$ is the factor by which the value $W(t)$ has to be multiplied to obtain the value at the end of the period, defined as the *capitalized value.* Conversely, $v(t, t+\Delta t)$ is the amount by which the value at the end of the period $W(t+\Delta t)$ needs to be multiplied to obtain the value at the beginning of the period, i.e. the *discounted value* or *present value* if time t can be interpreted as the actual time instant.
Notice that the following relations hold:

$$
\begin{aligned}
j(t, t+\Delta t) &= m(t+\Delta t) - 1; \\
d(t, t+\Delta t) &= 1 - v(t, t+\Delta t); \\
m(t, t+\Delta t) &= \frac{1}{v(t, t+\Delta t)}; \\
d(t, t+\Delta t) &= 1 - v(t, t+\Delta t) = m(t, t+\Delta t)v(t, t+\Delta t) - v(t, t+\Delta t) = \\
&= v(t, t+\Delta t)\left(m(t, t+\Delta t) - 1\right) = v(t, t+\Delta t)j(t, t+\Delta t).
\end{aligned}
$$

From this last relation, we have that

$$
\begin{aligned}
d(t, t+\Delta t) = v(t, t+\Delta t)\frac{\Delta W(t)}{W(t)} \\
W(t)d(t, t+\Delta t) = v(t, t+\Delta t)\Delta W(t).
\end{aligned}
\tag{1.2}
$$

From this last formula (1.2), it is evident that the discounted value of interests (the product on the right hand side) is equal to the interests, calculated using the anticipated rate of return (product on the left hand side of the equation).

Example 1.7. Let us consider a value function such that $W(t) = 95, W(t+\Delta t) = 100$. Then:

$$
\begin{aligned}
\Delta W(t) = 100 - 95 = 5; \\
j(t, t+\Delta t) = \frac{\Delta W(t)}{W(t)} = \frac{5}{95} = 5.26\%;
\end{aligned}
$$

$$d(t, t+\Delta t) = \frac{\Delta W(t)}{W(t+\Delta t)} = \frac{5}{100} = 5\%;$$

$$m(t, t+\Delta t) = \frac{W(t+\Delta t)}{W(t)} = \frac{100}{95} = 1.0526;$$

$$v(t, t+\Delta t) = \frac{W(t)}{W(t+\Delta t)} = \frac{95}{100} = 0.95.$$

Notice that the relations we highlighted above obviously hold:

$$W(t+\Delta t) = 100 = W(t)m(t, t+\Delta t) = 95 * 1.0526 = 100;$$

$$W(t) = 95 = W(t+\Delta t)v(t, t+\Delta t) = 100 * 0.95 = 95;$$

$$m(t, t+\Delta t) = 1.0526 = \frac{1}{v(t, t+\Delta t)} = \frac{1}{0.95} = 1.0526;$$

$$j(t, t+\Delta t) = 0.0526 = m(t, t+\Delta t) - 1 = 1.0526 - 1 = 0.0526;$$

$$d(t, t+\Delta t) = 0.05 = 1 - v(t, t+\Delta t) = 1 - 0.95 = 0.05 =$$

$$= v(t, t+\Delta t)j(t, t+\Delta t) = 0.95 * 0.0526 = 0.05.$$

1.7.1 Average rate of return and force of interest

So far, the length of the period over which interests are computed did not enter the picture. We now define:

- $\gamma(t, t+\Delta t) = \frac{j(t,t+\Delta t)}{\Delta t} = \frac{\Delta W(t)}{\Delta t W(t)}$ = the *average rate of return.*

Example 1.8. Consider $j(t, t+\Delta t) = 4\%, \Delta t = 2$ years. Find $\gamma(t, t+\Delta t)$.

$$\gamma(t, t+\Delta t) = \frac{4\%}{2} = 0.02 \text{ years}^{-1}.$$

Notice that the average rate of return is a "per year" (more in general, a "per unit of time") quantity. Due to this fact, it may be also referred to as to the *interest rate intensity.*
Analogously, the average anticipated rate of return is obtained dividing $d(t, t+\Delta t)$ by Δt.

If the value function W satisfies minimum regularity conditions at t, i.e. it is differentiable (has a first order time-derivative, $W'(t)$, or, in other words, its left-derivative and its right-derivative are equal), then we can define the *force of interest* or *instantaneous interest rate* or *instantaneous interest intensity*, as

$$\lim_{\Delta t \to 0^+} \frac{j(t, t+\Delta t)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{\Delta W(t)}{\Delta t W(t)} = \frac{W'(t)}{W(t)} = \frac{d \log W(t)}{dt} = \delta(t).$$

It represents the sensitivity of the value function to time variations, as a percentage change per unit of time; indeed, it is a so-called semi-elasticity.
Notice also that it coincides with the limit when Δt tends to 0^+ of the anticipated average rate of return:

$$\lim_{\Delta t \to 0^+} \frac{d(t, t+\Delta t)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{\Delta W(t)}{\Delta t W(t+\Delta t)} = \frac{1}{W(t)} \lim_{\Delta t \to 0^+} \frac{\Delta W(t)}{\Delta t} \frac{W(t)}{W(t+\Delta t)} =$$

$$= \frac{1}{W(t)} \lim_{\Delta t \to 0^+} \frac{\Delta W(t)}{\Delta t} 1 = \frac{W'(t)}{W(t)} = \delta(t).$$

Remark 1.1. To be able to compute $\delta(t)$ it is necessary to have the knowledge of $W(t)$ in a right neighborhood of t. Hence, it is not enough to know the value of the function at some points, it is necessary to know its behavior as well, in at least a small interval $[t, t+\Delta t]$.

Let us now consider the exponential law, $W(t) = W(0)(1+i_c)^t, i_c > -1$ and compute its force of interest:

$$\delta(t) = \frac{W'(t)}{W(t)} = \frac{W(0)(1+i_c)^t \log(1+i_c)}{W(0)(1+i_c)^t} = \log(1+i_c) = \delta.$$

Thus, the exponential law has a constant force of interest.

Remark 1.2. We can prove that the exponential law is the only law with a constant force of interest. Let us consider a differentiable and positive value function W with a constant force of interest, i.e. whose $\delta(t) = \frac{W'(t)}{W(t)} = \frac{d \log W(t)}{dt} = \delta, \forall t$. Then, the derivative of the log of W with respect to time is a constant. We can write the equality between the integrals of $\frac{dlogW(u)}{du}$ and δ over the interval $[0, t]$:

$$\int_0^t \frac{d \log W(u)}{du} du = \int_0^t \delta du = \delta t,$$

$$logW(t) - logW(0) = \delta t \implies log\frac{W(t)}{W(0)} = \delta t \implies \frac{W(t)}{W(0)} = e^{\delta t}$$
$$\implies W(t) = W(0)e^{\delta t}.$$

Denoting $e^{\delta} = (1 + i)$, i.e. $i = e^{\delta} - 1$, we obtain the usual expression of the exponential law, $W(t) = W(0)(1+i)^t$. Hence, it follows that any law with constant force of interest is an exponential law.

Also notice that it is useful to make the following rewriting:

$$W(t) = W(0)(1 + i_c)^t = W(0)e^{t\log(1+i_c)} = W(0)e^{\delta t}, \tag{1.3}$$

which holds true provided that $i_c > -1$.
The linear law, instead, does not have a constant instantaneous interest rate. Since $W(t) = W(0)(1 + i_s t)$, it follows that

$$\delta(t) = \frac{W'(t)}{W(t)} = \frac{W(0)i_s}{W(0)(1 + i_s t)} = \frac{i_s}{1 + i_s t},$$

which is a (decreasing, when $i_s > 0$), function of time.
From (1.3), the capitalization factor in the exponential can be expressed in terms of the force of interest δ as $e^{\delta \Delta t}$, whereas the discount factor is $e^{-\delta \Delta t}$. We refer to $u = e^{\delta} = (1 + i) = 1/v$ as the *annual capitalization factor*, where v is the *annual discount factor*, i.e. the discount factor computed for a unitary time interval.
Let us now consider the time - T value of a sum x_t at time t. This sum can be seen as $x_T = W(t)e^{\delta(T-t)}$, i.e. $W(t)$, the value at time t of the sum x_T, is $W(t) = x_T e^{-\delta(T-t)}$. These expressions give us the analogues of the expressions we obtained expressing the capitalization factor of the exponential law in terms of the

interest rate. Notice also that the sum x_T can be seen as

$$x_T = x_0 e^{\delta T} = \underbrace{x_t e^{-\delta t}}_{x_0} e^{\delta T} = x_t e^{\delta(T-t)}.$$

Summarizing, having x_T at T is equivalent to have either $x_0 = x_T e^{-\delta T}$ at 0 or to $x_t = x_T e^{-\delta(T-t)}$ at t. Indeed, the analogous equivalence can be stated in terms of the expressions involving the interest rate: the sum x_T in T is equivalent to $x_0 = x_T(1+i)^{-T}$ at 0 and to $x_t = x_T(1+i)^{-(T-t)}$ at time t.
Notice that all the fundamental quantities we have defined above can be written, for the exponential law, in terms of the force of interest δ. Indeed:

- $W(t+\Delta t) = W(t)(1+i)^{\Delta t} = W(t)e^{\delta \Delta t}$;
- $\Delta W(t) = W(t+\Delta t) - W(t) = W(t)(1+i)^{\Delta t} - W(t) = W(t)e^{\delta t} - W(t) = W(t)\left[e^{\delta t} - 1\right]$;
- $m(t, t+\Delta t) = \frac{W(t+\Delta t)}{W(t)} = (1+i)^{\Delta t} = e^{\delta \Delta t}$;
- $v(t, t+\Delta t) = \frac{W(t)}{W(t+\Delta t)} = e^{-\delta \Delta t}$;
- $j(t, t+\Delta t) = m(t, t+\Delta t) - 1 = (1+i)^{\Delta t} - 1 = e^{\delta \Delta t} - 1$;
- $d(t, t+\Delta t) = 1 - v(t, t+\Delta t) = 1 - (1+i)^{-\Delta t} = 1 - e^{-\delta \Delta t}$;
- $\gamma(t, t+\Delta t) = \frac{j(t,t+\Delta t)}{\Delta t} = \frac{(1+i)^{\Delta t}-1}{\Delta t} = \frac{e^{\delta \Delta t}-1}{\Delta t}$.

Remark 1.3. All the above quantities, apart from the interest, do not depend on the specific point in time in which they are computed, but only on the length of the time interval of interest. Hence, unitary quantities do not change in time. This is evidently a consequence of the constant nature of the force of interest.

Example 1.9. Consider an exponential law with annual compound interest rate $i = 3\%$. Compute the periodic interest rates $j(1,3)$ and $j(3,5)$ and then all the fundamental quantities relative to the period $[3,5]$.

$$j(1,3) = (1+3\%)^2 - 1 = 1.0609 - 1 = 0.0609;$$

$$j(3,5) = (1+3\%)^2 - 1 = 1.0609 - 1 = 0.0609;$$

$$\delta = \log(1+i) = 5.9112\%;$$

$$m(3,5) = 1 + j(3,5) = 1.0609;$$

$$v(3,5) = \frac{1}{m(3,5)} = 0.9426;$$

$$d(3,5) = 1 - v(3,5) = 1 - 0.9426 = 0.0574.$$

1.8 Equivalent rates

1.8.1 Time measurement standards

Time can be measured according to different standards. The length of time intervals can be measured in several ways, that always have to be contractually specified. In particular, the time distance between two dates can be measured:

- in effective (actual) calendar days, counting the number of days between the two dates and excluding either the first or the last day;
- according to the 30-day per month convention, that assumes that all months have 30 days.
- considering the year formed by 365 days always, or 366 days in case of a leap year (actual method), or by 360 days (30-day method).

The way time is measured is defined by a fraction that defines the way days are counted at the numerator and the number of days in one year at the denominator. EFF (ACT) is used as a shortcut for effective. ACT used at the denominator defines the year with its effective number of days, i.e. 365 or 366 depending on the number of days in the current year. $\frac{30}{360}$ is used when days are counted following the 30-day rule and it is assumed that a year is made of 360 days. Interests computed following such convention are termed *ordinary interests*. Interests computed following the $\frac{ACT}{360}$ convention are said to follow the *Banker's rule*.

Example 1.10. Let us consider a financial contract valued 97.8 euros at $t = 0$ and 101.5 after 95 days. Compute: the interest, the rate of return, the anticipated rate

of return, the average rate of return. Considering an exponential law underlying the contract, and measuring the year in effective days, compute: the annual compound interest rate, the annual instantaneous interest rate.
Let us denote with $W(0) = 97.8$ the value of the contract at 0 and with $W(95) = 101.5$ its value at 95 days. Then:

$$\Delta W(t) = 101.5 - 97.8 = 3.7;$$

$$j(0, 95) = 3.7/97.8 = 3.783\%$$

$$d(0, 95) = 3.7/101.5 = 3.645\%;$$

$$\gamma(0, 95) = \frac{0.03783}{95 \text{ days}} = 0.000398;$$

$$W\left(\frac{95}{365}\right) = W(0)(1 + i)^{\frac{95}{365}} \implies i = \left(\frac{W(\frac{95}{365})}{W(0)}\right)^{\frac{365}{95}} - 1 = 15.34\%;$$

$$\delta = log(1 + i) = 0.1426.$$

1.8.2 Equivalent Rates

It is often necessary, for practical reasons, to compare financial transactions relative to different time intervals or time lengths, or where time is measured according to different standards. There is then a need to tackle the issue of transforming interest rates expressed in one time unit of measure to another one. The objective is then to identify the parameters that imply the equivalence between the same financial law defined on different units of measure of time.
Let us fix, for instance, an exponential law whose annual compound rate is i. We may want to define, for time intervals shorter than the year, the *equivalent rate* as the periodic rate relative to that time interval that identifies the same financial law. Let us define as i_q the periodic rate, where $1/q, q > 1$ defines the length of the time interval of interest relative to the starting unitary time measure (the year). We are indeed considering a change in the unit of measure of time: q defines the number of units of time in the "new" measure for each "old" unit of measure. For instance, $q = 2$ defines the half-year, $q = 4$ the quarter. Then, to find the equivalent rate i_q, we need to find the rate according to which the same financial law is defined before and after the change of measure. In other words, considering a law according to which $W(t) = W(0)(1 + i)^t$, with t measured in a certain unit of measure, we have to find the i' such that

$$W(0)(1 + i)^t = W(0)(1 + i_q)^{t'},$$

where t' is the time measured in the "new" unit of measure, i.e. $t' = tq$. Let us consider the exponential law case first. Then, we need to find the rate i_q that solves

$$W(0)(1 + i)^t = W(0)(1 + i_q)^{t'} = W(0)(1 + i_q)^{tq}.$$

Indeed:

$$(1 + i) = (1 + i_q)^q \implies i_q = \sqrt[q]{1 + i} - 1.$$

Example 1.11. Find the quarterly interest rate i_4 equivalent to the annual compound interest rate $i = 5\%$.

$$(1+5\%) = (1+i_4)^4 \implies i_4 = \sqrt[4]{1.05} - 1 = 0.0123.$$

If we consider the linear law, we have that

$$W(0)(1+it) = W(0)(1+i_q t') = W(0) = 1 + i_q tq \implies i_q = \frac{i}{q}.$$

Example 1.12. Find the monthly interest rate, i_{12}, equivalent to the annual simple interest rate $i = 3\%$.

$$1 + 3\% = 1 + i_p 12 \implies i_p = \frac{3\%}{12} = 0.25\%.$$

The same reasoning applies to the other unitary quantities. Let us focus on the exponential law. Indeed, for the instantaneous interest rate intensity δ we have that:

$$\delta = log(1+i) = log[(1+i_q)^q] = qlog(1+i_q) = q\delta_q.$$

It then follows that:

$$u = (1+i) = (1+i_q)^q = (u_q)^q;$$
$$v = \frac{1}{u} = \frac{1}{(1+i)} = \frac{1}{(1+i_q)^q} = (v_q)^q$$
$$d = 1 - v = 1 - (v_q)^q = (1 - (1-d_q)^q) \implies 1 - d = (1-d_q)^q.$$

Example 1.13. Let us compute the quarterly equivalent force of interest, δ_4, u_4, v_4, d_4 of the exponential law with annual compound interest rate $i = 5\%$.
First of all, we have to compute $\delta = \log(1+5\%) = \log 1.05 = 0.0488$. Now, as $q = 4$, δ_q will be simply equal to $\delta/4 = \frac{0.0488}{4} = 0.0122$. Also, we have that:

$$u = e^{\delta} = 1.05 \implies u_4 = \sqrt[4]{1.05} = 1.0123;$$

$$v = e^{-\delta} = 0.9523 \implies v_4 = \sqrt[4]{0.9523} = 0.9879;$$

$$d = 1 - v = 0.0477 \implies d_4 = 1 - \sqrt[4]{1-d} = 0.0121.$$

1.9 Financial contracts: a more general representation.

So far, we focused on elementary financial transactions, i.e. contracts that involve the exchange of an inflow (outflow, depending on the point of view) at a certain instant t and an outflow (inflow) at another time instant $t + \Delta t$. We refer to such contract as to elementary financial contracts. More in general, when the contract rules the exchange of $n+1$ amounts at $n+1$ dates, it is represented as the sequence of amounts $\mathbf{x} = \{x_0, x_1, ..., x_n\}$ and dates at which the amounts are exchanged $\mathbf{t} = \{t_0, t_1, ..., t_n\}$, with $t_0 < t_1 < ... < t_n$, referring to it using the notation $\mathbf{x}/\mathbf{t}$. As usual, $x_k > 0$ denotes an inflow of amount $|x_k|$, $x_k < 0$ denotes an outflow $|x_k|$.

1.9.1 Algebraic operations between financial contracts

First of all, we remark that a financial transaction $\mathbf{x}/\mathbf{t}$ always coincides with a financial transaction having the same amounts and dates and additional exchanges of zero amount at any additional possible date.

Example 1.14. The transaction $\{-100, 50, 120\}/\{0, 1, 2\}$ is equivalent to the transaction $\{-100, 0, 50, 0, 120\}/\{0, 0.5, 1, 1.5, 2\}$.

It is then possible to define the sum of two financial contracts. Given two contracts, $\mathbf{x'}/\mathbf{t'}$ and $\mathbf{x''}/\mathbf{t''}$, their sum is defined as the financial contract $\mathbf{x}/\mathbf{t}$, where $\mathbf{t} = \mathbf{t'} \cup \mathbf{t''}$ denotes the union of the dates at which exchanges occur in the two contracts and the amounts $\mathbf{x}$ are obtained adding the amounts in the two contracts at each date, completing $\mathbf{x'}$ and $\mathbf{x''}$ with null amounts at the additional dates in $\mathbf{t}$.

Example 1.15. Consider the contracts $\{-100, 40, 60\}/\{0, 1, 2\}$ and $\{50, -30, 100\}$ / $\{0, 0.5, 1.5\}$. Their sum is given by the contract whose set of dates is $\{0, 0.5, 1, 1.5, 2\}$, and whose amounts are $\{-50, -30, 40, 100, 60\}$.

We can also define the product of $\mathbf{x}/\mathbf{t}$ times a real number $\alpha \in \mathbb{R}$, as the contract $\alpha\mathbf{x}/\mathbf{t}$.

Example 1.16. Consider the product $3 * \{-100, 40, 60\}/\{0, 1, 2\}$. This is nothing but the contract $\{-300, 120, 180\}/\{0, 1, 2\}$.

1.9.2 Decomposition of financial transactions

Financial transactions can be also decomposed, by considering them as the sum of two or more financial transactions identified as subsets of elements in the vectors of amounts and dates. In particular, it is useful to mention two notable decompositions: the infows/outflows decomposition and the past/future decomposition. The former obtains the financial contract as the sum of two transactions: one in which only the inflows are considered and one in which all the outflows are grouped. The latter obtains the financial contract as the sum of two transactions: one in which all the cash flows occurring prior to a certain instant t (the present) are considered and one in which all the cash flows occurring after t are grouped. More formally, we define the inflow transaction as $\mathbf{i}/\mathbf{t}$, where $\mathbf{i} = \{i_0, i_1, ..i_n\}$, with $i_k = \begin{cases} x_k \text{ if } x_k \geq 0 \\ 0 \text{ if } x_k < 0 \end{cases}$ and the outflow transaction $\mathbf{o}/\mathbf{t}$, where $\mathbf{o} = \{o_0, o_1, ..o_n\}$, with $o_k = \begin{cases} x_k \text{ if } x_k < 0 \\ 0 \text{ if } x_k \geq 0 \end{cases}$. Also, having fixed a time instant t, we define the "accumulated" transaction as $\mathbf{a}/\mathbf{t}$, where $\mathbf{a} = \{a_0, a_1, ..a_n\}$, with $a_k = \begin{cases} x_k \text{ if } t_k \leq t \\ 0 \text{ if } t_k > t \end{cases}$ and the "residual" transaction $\mathbf{r}/\mathbf{t}$, where $\mathbf{r} = \{r_0, r_1, ..., r_n\}$, with $f_k = \begin{cases} x_k \text{ if } t_k > t \\ 0 \text{ if } t_k \leq t \end{cases}$. Notice that, by convention, amounts exchanged at $t = t_k$ belong to the "past", i.e. are considered in the "accumulated" transaction.

Example 1.17. Consider the financial transaction $\mathbf{x}/\mathbf{t}$, $\{-1000, 300, 500, 700\}$ / $\{0, 1, 2, 3\}$. It can be decomposed into the transaction that considers only its outflows (liabilities) $\mathbf{o}/\mathbf{t}$, $\{-1000, 0, 0, 0\}/\{0, 1, 2, 3\}$ and its inflows (assets) $\mathbf{i}/\mathbf{t}$, $\{0, 300, 500, 700\}$ / $\{0, 1, 2, 3\}$: $\mathbf{x}/\mathbf{t} = \mathbf{i}/\mathbf{t} + \mathbf{o}/\mathbf{t}$. Fixing the time instant 1.5 as the reference, we can decompose it into the sum of the two transactions containing past flows, $\mathbf{a}/\mathbf{t}$, $\{-1000, 300, 0, 0\}/\{0, 1, 2, 3\}$, and future flows of money, $\mathbf{r}/\mathbf{t}$, $\{0, 0, 500, 700\}/\{0, 1, 2, 3\}$: $\mathbf{x}/\mathbf{t} = \mathbf{a}/\mathbf{t} + \mathbf{r}/\mathbf{t}$.

1.10 Evaluating a financial transaction under the exponential law

Let us consider a financial transaction $\mathbf{x}/\mathbf{t}$, with the first element of the $\mathbf{t}$ vector, t_0, positive, without loss of generality and a given exponential law with interest rate i (or, equivalently, with instantaneous interest rate intensity δ).
Fixing a certain valuation instant t, with $t \geq t_0$, we define as $W(t, x_k)$ the value of the sum x_k, exchanged at time t_k, at time t. It is equal to:

$$W(t, x_k) = x_k e^{\delta(t-t_k)} = x_k(1+i)^{t-t_k}.$$

Definition 1.1. We define the value of a financial contract $\mathbf{x}/\mathbf{t}$ at time t according to an exponential law with annual compound interest rate i (or, equivalently, instantaneous interest rate δ) the value $W(t, \mathbf{x}/\mathbf{t})$ obtained as:

$$W(t, \mathbf{x}/\mathbf{t}) = \sum_{k=0}^{n} W(t, x_k) = \sum_{k=0}^{n} x_k e^{\delta(t-t_k)} = \sum_{k=0}^{n} x_k(1+i)^{t-t_k}.$$

Notice that all the sums being exchanged prior to the valuation instant t are capitalized up to t using the appropriate accumulation factor, whereas sums being exchanged after the valuation instant are discounted using the appropriate discount factor. Notice also that the value of the contract depends on the time t at which the contract itself is evaluated.

Example 1.18. Consider the following financial contract: $\mathbf{x}/\mathbf{t} = \{-1000, 200, 1500\}$ / $\{0, 1.5, 2\}$. Let us compute its value at $t = 1$, under an exponential law with annual compound interest rate $i = 2\%$. It is equal to:

$$W(1, \mathbf{x}/\mathbf{t}) = -1000(1.02)^1 + 200(1.02)^{-0.5} + 1500(1.02)^{-1} =$$

$$= -1020 + 198.03 + 1470.59 = 648.72.$$

Its value at $t = 0.5$ is indeed:

$$W(0.5, \mathbf{x}/\mathbf{t}) = -1000(1.02)^{0.5} + 200(1.02)^{-1} + 1500(1.02)^{-1.5} =$$

$$= -1009.95 + 196.08 + 1456.10 = 642.23.$$

1.11 Financial fairness of a financial transaction

Definition 1.2. Given a certain exponential law, a financial operation $\mathbf{x}/\mathbf{t}$ is *fair* according to that financial law at time t if

$$W(t, \mathbf{x}/\mathbf{t}) = 0.$$

It can be easily noticed that the definition of *fair* transaction implies that for such transaction the value of outflows is equal to the value of inflows. Indeed, notice that, indicating as in the previous section with $\mathbf{i}/\mathbf{t}$ and $\mathbf{o}/\mathbf{t}$ the financial transactions generating the assets/liabilities decomposition of $\mathbf{x}/\mathbf{t}$, financial fairness implies

$$W(t, \mathbf{x}/\mathbf{t}) = W(t, \mathbf{i}/\mathbf{t}) + W(t, \mathbf{o}/\mathbf{t}) = 0.$$

As a corollary, notice that, in order to be fair, a financial transaction needs at least one inflow and one outflow.
Referring instead to the residual vs. accumulated flows decomposition, we will call *accumulated value* value of $\mathbf{x}/\mathbf{t}$ at t, and refer to it as to $M(t, \mathbf{x}/\mathbf{t})$, the capitalized value at time t of the "accumulated" transaction of the accumulated vs. residual decomposition of $\mathbf{x}/\mathbf{t}$, i.e. the sum of the capitalized values of the inflows/outflows occurring prior to time t:

$$M(t, \mathbf{x}/\mathbf{t}) = W(t, \mathbf{a}/\mathbf{t}) = \sum_{k:t_k \leq t} x_k e^{\delta(t-t_k)}$$

Analogously, we define the *residual value* of $\mathbf{x}/\mathbf{t}$ at time t as the time-t value of the "residual" transaction, i.e. the sum of the discounted value of the inflows/outflows occurring after time t:

$$V(t, \mathbf{x}/\mathbf{t}) = W(t, \mathbf{r}/\mathbf{t}) = \sum_{k:t_k > t} x_k e^{-\delta(t_k - t)}.$$

Notice that $V(t_n, \mathbf{x}/\mathbf{t}) = 0$. Notice also that, if the transaction is fair at time t, then:

$$M(t, \mathbf{x}/\mathbf{t}) = -V(t, \mathbf{x}/\mathbf{t}).$$

A recursive equation regarding the accumulated value can be obtained. Denote for simplicity $M_k = M(t_k, \mathbf{x}/\mathbf{t})$. For $k = 1, 2, ..., n$, M_k is equal to

$$\begin{aligned} M_k = \sum_{h:h\leq k} x_h e^{\delta(t_k - t_h)} = \\ = x_k + \sum_{h:h<k} x_h e^{\delta(t_k - t_h)} = \\ = x_k + e^{\delta(t_k - t_{k-1})} \underbrace{\sum_{h:h\leq k-1} x_h e^{\delta(t_{k-1} - t_h)}}_{M_{k-1}}. \end{aligned}$$

Hence, we can finally write

$$M_k = x_k + M_{k-1} e^{\delta(t_k - t_{k-1})} = x_k + M_{k-1}(1+i)^{t_k - t_{k-1}}, k = 1, 2, ..., n.$$

1.11.1 Properties of the exponential law

The exponential law enjoys 4 functional properties:

A *Invariance*: if a financial transaction is fair at time t according to a fixed exponential law, it is also fair at any other instant;

B *Additivity*: if two financial transactions are fair at time t, according to a certain exponential law, then also their sum is fair at t;

C *Time uniformity*: if a financial transaction is fair at time t according to a certain exponential law, then also the transaction that has same amounts and all the dates shifted by a factor Δt is fair at $t + \Delta t$.

D *Decomposability*: the sum of two transactions, fair at distinct time instants, according to a certain financial law, is a fair transaction at any time instant.

Proof. A - Fairness at time t implies that

$$W(t, \mathbf{x}/\mathbf{t}) = 0 \implies \sum_{x=0}^{k} x_k e^{\delta(t - t_k)} = 0.$$

Let us consider a different time instant $t' = t + \Delta t > 0$, $\Delta t \in \mathbb{R}$. Then:

$$W(t', \mathbf{x}/\mathbf{t}) = \sum_{x=0}^{k} x_k e^{\delta(t+\Delta t - t_k)} =$$

$$= \sum_{x=0}^{k} x_k e^{\delta(t-t_k)} e^{\delta \Delta t} = e^{\delta \Delta t} \sum_{x=0}^{k} x_k e^{\delta(t-t_k)} = e^{\delta \Delta t} W(t, \mathbf{x}/\mathbf{t}) = 0.$$

B - Let us consider $\mathbf{x}/\mathbf{t}$ and $\mathbf{y}/\mathbf{t}$ both fair at time t and defined on the common set of dates $\mathbf{t}$. Then, fairness at t implies that

$$W(t, \mathbf{x}/\mathbf{t}) = W(t, \mathbf{y}/\mathbf{t}) = 0.$$

$$W(t, \mathbf{x}+\mathbf{y}/\mathbf{t}) = \sum_{k=0}^{n} (x_k + y_k) e^{\delta(t-t_k)} = \sum_{k=0}^{n} x_k e^{\delta(t-t_k)} + \sum_{k=0}^{n} y_k e^{\delta(t-t_k)} =$$
$$= W(t, \mathbf{x}/\mathbf{t}) + W(t, \mathbf{y}/\mathbf{t}) = 0.$$

C - Fairness at time t implies that

$$W(t, \mathbf{x}/\mathbf{t}) = 0 \implies \sum_{x=0}^{k} x_k e^{\delta(t-t_k)} = 0.$$

Now, consider the financial transaction $\mathbf{x}/\mathbf{t} + \Delta t$, where for every k $t_k = t_k + \Delta t$. Then, the value of such transaction at $t + \Delta t$ is equal to

$$W(t + \Delta t, \mathbf{x}/\mathbf{t} + \Delta t) = \sum_{x=0}^{k} x_k e^{\delta(t+\Delta t-(t_k+\Delta t))} = \sum_{x=0}^{k} x_k e^{\delta(t-t_k)} = W(t, \mathbf{x}/\mathbf{t}) = 0.$$

D - This property descends directly from A and B. □

Properties C and D can be immediately transposed into analogous properties of the discount factors. Let us consider three time instants, $t_0 \leq t_1 \leq t_2$. Then, decomposability implies that:

$$v(t_0, t_2) = v(t_0, t_1) v(t_1, t_2).$$

This property implies that discounting over an interval is always equivalent to repeated discounting over adjacent sub-intervals.

Time uniformity implies that, given $\Delta t \geq 0$:

$$v(t, t + \Delta t) = v(\Delta t).$$

In general, financial laws whose discount factors satisfy the above equation are said to be uniform. Indeed, in a uniform financial law, the discount factor depends only on the length of the discounting period and not on the initial or final dates defining the discounting period.
It can be shown that the exponential law is the only financial law that is simultaneously time uniform and decomposable.

Example 1.19. Let us consider an exponential law with annual compound interest rate $i_c = 3\%$. Then, it is easy to see that

$$v(0, 2) = (1 + i_c)^{-2} = 1.03^{-2} = 0.9426;$$
$$v(0, 1.5) = (1 + i_c)^{-1.5} = 1.03^{-1.5} = 0.9566;$$
$$v(1.5, 2) = (1 + i_c)^{-0.5} = 1.03^{-0.5} = 0.9853.$$

$$v(0, 2) = (1 + i_c)^{-2} = v(0, 1.5)v(1.5, 2) = (1 + i_c)^{-1.5}(1 + i_c)^{-0.5} = 0.9426.$$

As a counter-example, consider a linear law with annual simple interest rate $i_s = 3\%$. Then, it is easy to show that the linear law is not decomposable. Indeed, we have:

$$v(0, 2) = \frac{1}{1 + i_s * 2} = 0.9434;$$
$$v(0, 1.5) = \frac{1}{1 + i_s * 1.5} = 0.9569;$$
$$v(1.5, 2) = \frac{1}{1 + i_s * 0.5} = 0.9852.$$

$$v(0, 1.5)v(1.5, 2) = \frac{1}{1 + i_s * 1.5} * \frac{1}{1 + i_s * 0.5} = 0.9427 \neq \frac{1}{1 + i_s 2} = v(0, 2) = 0.9434.$$

Notice also that both the exponential law and the linear law are uniform, as the discount factors $v(t, t+\Delta t)$depend on Δt only and not on t and $t+\Delta t$.

1.11.2 A few more remarks on financial fairness

Remark 1.4. Any financial transaction can be modified to be fair, by adding a cash flow y_t at time t equal to minus the time-t value of the financial transaction, i.e. $y_t = -W(t, \mathbf{x}/\mathbf{t})$. Indeed, calling $\mathbf{y}/\mathbf{t}$ the modified transaction:

$$W(t, \mathbf{y}/\mathbf{t}) = y_t + W(t, \mathbf{x}/\mathbf{t}) = -W(t, \mathbf{x}/\mathbf{t}) + W(t, \mathbf{x}/\mathbf{t}) = 0.$$

Example 1.20. Consider the transaction $\mathbf{x}/\mathbf{t}$ with $\mathbf{x} = \{0, 3, 3, 103\}$ and $\mathbf{t} = \{0, 1, 2, 3\}$. Considering an exponential law with annual compound rate $i = 3\%$, its time 0-value $W(0, \mathbf{x}/\mathbf{t}) = 100$. So, the transaction is not fair. Let us now consider $\mathbf{y}/\mathbf{t}$, where $\mathbf{y} = \{-W(0, \mathbf{x}/\mathbf{t}), 3, 3, 103\}$. Then, $\mathbf{y}/\mathbf{t}$ is fair, by construction.

Remark 1.5. Fairness is not invariant to changes in the underlying financial law used to evaluate the financial transaction.

Example 1.21. Consider the above transaction $\mathbf{y}/\mathbf{t}$, fair when $i = 3\%$. If $i = 4\%$, then:

$$W(0, \mathbf{y}/\mathbf{t}) = -100 + 3 * 1.04^{-1} + 3 * 1.04^{-2} + 103 * 1.04^{-3} = -2.77.$$

The value at the final date 3 is equal to:

$$W(3, \mathbf{y}/\mathbf{t}) = -100 * 1. - 03^3 + 3 * 1.04^2 + 3 * 1.04^1 + 103 = -3.12.$$

Notice that, due to decomposability, we can use our knowledge of $W(0, \mathbf{y}/\mathbf{t})$ to compute $W(3, \mathbf{y}/\mathbf{t})$:

$$W(3, \mathbf{y}/\mathbf{t}) = W(0, \mathbf{y}/\mathbf{t}) m(0, 3) = -2.77 * 1.04^3 = -3.12.$$

1.12 Exercises

Exercise 1.1. Mr. Smith obtains a loan of 10000 euros from bank A, that he has to reimburse after one year 3 months, paying an interest of 840 euros. Compute the rate of return j for the whole period and the annual compound interest rate i_c of the associated exponential law. Bank B offers the same loan, for the same period, charging an annual simple interest $i_s = 7.1\%$. Compute the rate of return j' relative to the whole period and the annual compound interest rate i_c that would apply if the bank was charging a compound interest rate with the same interest implied by its linear law. Which loan should Mr. Smith take?

Exercise 1.2. Find the times T and T' at which an initial invested capital S doubles, according to an exponential law with $i_c = 3\%$ and a linear law with $i_s = 3\%$, respectively.

Exercise 1.3. Bank Red proposes a loan of 15000 euros to an entrepreneur, with the agreement that she will reimburse the loan after T years, together with an interest of 5000 euros. Find T, knowing that the interest is charged at an annual compound rate $i = 4\%$. Bank Green offers to loan the same amount for T' years, charging an annual simple interest $i' = 5\%$. Find T' such that the entrepreneur will reimburse the same amount reimbursed to Bank Red. Which of the two offers is more convenient to the entrepreneur?

Exercise 1.4. An initial capital S euros is invested for 3 years and 8 months at the annual simple interest $i = 6.46\%$, obtaining an interest equal to 975 euros. Find S.

Exercise 1.5. Mr. Smith has two alternative investment options:

- Account Green, where the invested capital grows at an annual compound rate $i_G = 3\%$;
- Account Red, where the invest capital grows at an annual simple rate $i_R = 4\%$. Mr. Smith wants to invest until its initial sum has grown by 1/4. Determine the holding periods T_G and T_R necessary to reach Mr. Smith's objective in the two cases and the interest rates j_G and j_R relative to the periods $[0, T_G]$ and $[0, T_R]$ respectively. Which investment option should Mr. Smith choose?

1.12.1 Addendum: compounding at different intervals

In the previous section we assumed that interests were calculated at the end of each period (year) and paid at that time. However, it is common practice for banks to compute the interests at a higher frequency (usually quarterly, but some times even monthly or daily).
A more frequent compounding raises the so-called *effective yearly rate*. Suppose for instance that a yearly interest rate i is compounded quarterly, i.e. an interest rate of $i/4$ is applied every three months. For any $i > 0$ this implies that, at the end of the year a unit account will grow to $(1 + i/4)^4 > (1 + i)$. $(1 + i/4)^4 - 1$ is referred to as the *effective interest rate*, whereas i is termed the *yearly nominal rate*.
A passage to the limit allows us to define the *continuously compounded interest rate*. Let $1/m$ be the compounding frequency, where m is the number of equally spaced periods in which a period of unitary length is divided into for compounding reasons. Let m grow indefinitely. Then, the time-t value of 1 euro in 0 will be:

$$\lim_{m\to\infty}\left[\left[1+\frac{i}{m}\right]^m\right]^t = e^{it},$$

which follows from the fact that $\lim_{m\to+\infty}\left(1+\frac{x}{m}\right)^m = e^x$. Thus, the effective interest rate i' satisfies $(1 + i')^t = e^{it}$.

2 Internal rate of return

The last sections of the previous chapter focused on how to evaluate a financial transaction according to a given exponential law. In particular, the present value of a transaction at time t can be seen as the time-t outflow that needs to be paid so that the transaction, together with such payment, is fair, given a certain exponential law, i.e. fixed a compound annual interest rate i. In this section, we tackle the inverse problem, that is finding the interest rate i of an exponential law that makes a given financial transaction fair. Such interest rate is called the *internal rate of return (IRR)*. In general, the problem of finding the IRR of a financial transaction may be not well-posed, because the IRR may not exist, not be unique or may not be financially meaningful. However, when it exists and it is unique, the IRR provides an easily interpretable and powerful synthetic indicator regarding the attractiveness of an investment, and is therefore a widely used measure.

2.1 The internal rate of return

Definition 2.1. Given a financial transaction $\{\mathbf{x}\}/\{\mathbf{t}\}$, we define the *internal rate of return* as the interest rate of the exponential law that makes the financial transaction fair. In other words, it is the solution i^* of the following equation in the unknown i:

$$\sum_{k=0}^{n} x_k(1+i)^{-(t_k-t_0)} = 0. \tag{2.1}$$

A few remarks are useful.

Remark 2.1. Equation (2.1) can be written as:

$$-x_0 = \sum_{k=1}^{n} x_k(1+i)^{-(t_k-t_0)} = P.$$

x_0 can thus be interpreted as the present value of the sum of all the cash flows occurring at the times t_k, with $k > 0$, evaluated according to an exponential law with rate i^*.

Remark 2.2. Due to the time uniformity property of the exponential law, we can impose $t_0 = 0$ in equation (2.1), and rewrite the equation as

$$\sum_{k=0}^{n} x_k v^{t_k} = 0,$$

where v is the annual discount factor $v = \frac{1}{1+i}$. Notice also that, after an appropriate change of unit of measure, it is always possible to consider the above equation as depending on a set of integer values $t_k, k > 0$, so that $t_k = k \in \mathbb{N}$.

Following this last remark, we can rewrite the equation (2.1) as

$$\sum_{k=0}^{n} x_k v^k = 0. \tag{2.2}$$

Example 2.1. Consider the financial transaction $\mathbf{x}/\mathbf{t}$, with $\mathbf{x} = \{-100, 51, 51\}$, $\mathbf{t} = \{0, 0.25, 0.5\}$. The internal rate of return can be computed solving the following equation:

$$\sum_{k=0}^{2} x_k v^k,$$

where v is the quarterly discount factor and $k = 0, 1, 2$ is the time in $\mathbf{t}$ expressed in quarters. Thus, i^* is obtained as the solution of the second order equation

$$-100 + 51v + 51v^2 = 0.$$

Its discriminant Δ is equal to $51^2 + 51 * 100 * 4 > 0$ and hence the equation admits two distinct real roots. v is obtained using the well known formula:

$$v = \frac{-51 \pm \sqrt{51^2 + 51 * 100 * 4}}{2 * 51},$$

and hence the two roots are 0.986871 and -202.61. The first root leads us to $i = 1/v - 1 = 0.0133$. The second root is not acceptable, because it would lead to $i < -1$ that cannot be the interest rate of the transaction we described. Hence, our IRR will be the annual compound interest rate equivalent to the i we have just found:

$$i^* = (1 + i)^2 - 1 = 2.68\%.$$

Remark 2.3. Since the fairness of $\mathbf{x}/\mathbf{t}$ given a certain financial law implies fairness of $-\mathbf{x}/\mathbf{t}$ as well, the IRR of these two transactions will be the same, i.e. IRR gives the same result indepedently of the nature of the transaction (e.g. investment vs. financing operation).

2.2 Existence and uniqueness of the IRR

The function appearing at the l.h.s. of (2.2) is an n-th degree polynomial in the variable v with coefficients x_k (eventually null). Hence, equation (2.2) is an algebraic equation of degree n in the unknown v. In general, such an equation admits up to n roots in $\mathbb{R}$ and exactly n roots in $\mathbb{C}$. Indeed, a polynomial of degree n always admits a unique factorization

$$\sum_{x=0}^{n} x_k v^k = x_n (v - r_1)^{n_1} (v - r_2)^{n_2} \cdots (v - r_h)^{n_h},$$

with $r_1, ..., r_h \in \mathbb{C}$ and $n_1, ..., n_h \in \mathbb{R}_+$ s.t. $n_1 + ... + n_h = n$. Hence, the equation (2.2) admits exactly n complex roots, counted with their with multiplicity and in particular, will have h complex roots $r_1, ..., r_h$ with multiplicity $n_1, ..., n_h$, respectively. If $h = n$, there will be n distinct roots. If $n_1 = n$ there will be a unique root r_1 with multiplicity n.
For our purpose, we are willing to consider equations that have real solutions, because complex solutions have no financial interepretation.

Example 2.2. Consider the contract $\{50, 50, 50\}/0, 1, 2$. The IRR can be found as the solution to the following second degree equation:

$$50 + 50v + 50v^2 = 0.$$

The discriminant of such second order equation is

$$\Delta = 1^2 - 4 = -3 < 0.$$

Hence, the IRR equation for this transaction has no real roots and the IRR method can not be used.

Also, we are seeking for $i^* > -1$, because $i^* < -100\%$ refers to transactions in which one party loses the whole invested capital, which is not a situation that

applies to our setting in which there is no risk.
There are also some cases in which real solutions exist, but the information they convey is not useful to determine the cost of the transaction.

Example 2.3. Consider the transaction $\{70, -150, 80\}/\{0, 1, 2\}$. In this case the discriminant is $\Delta = 150^2 - 70 * 80 * 4 = 100 > 0$ and thus two distinct real roots exist. These roots are 1 and 0.875 respectively, yielding $i_1^* = 0$, $i_2^* = 14.85\%$. and it is not clear which of the two solutions provide a correct representation of the worthiness of the transaction.

Hence, the most interesting case, in which the internal rate of return is a meaningful synthetic measure of the cost or yield of an investment, is one in which there is uniqueness of a positive solution in the interval $(0, 1)$. To characterize the cases in which positive solutions exist, it is useful to recall Descartes' rule of signs:

Theorem 2.1. Let N be the number of sign variations in the succession of coefficients of a polynomial equation and let h be the number of positive roots. Then, $N - h$ is either a positive even number or zero.

The theorem is useful to our purpose because it tells us that, as a corollary:

- if the cash flows of $\mathbf{x}$ all have the same sign, i.e. $N = 0$, there are no real and positive solutions to (2.2);
- if there is only one sign exchange in the succession of cash flows the two parties have clear and distinct roles (one can be seen as a debtor and the other as a creditor), i.e. $N = 1$, there exists a unique positive solution ($h = 1$).

Hence, we have found a characteristic of the sequence of cash flows that surely allows for a unique and positive solution to the IRR problem: the cash flows need to change sign at most once. Notice that when $N \geq 2$ it may be possible that 0, 1 or more solutions exist.
However, even uniqueness of a positive solution for v is still not enough to guarantee that the internal rate of return is positive, i.e. that $v^* < 1$.
Let us consider from now on financial contracts where all the outflows precede the inflows (or viceversa), that is with only one sign change, $N = 1$, and in

particular, without loss of generality, a contract that has $x_0 = -P, P > 0$ and $x_i \geq 0, i = 1, ...n-1$ and $x_n > 0$. Such a contract, for the payer of x_0 at time 0 is an investment reimbursted in a series of instalments.

Let us define $f(v) = \sum_{k=1}^{n} x_k v^k$. $f(v)$ can be interpreted as the value of the residual cash flows of the contract, after having paid the price P at time 0. When $v \geq 0$ we have:

$$f(0) = 0; f(1) = \sum_{k=1}^{n} x_k; \lim_{v \to \infty} f(v) = +\infty.$$

Also, $f(v)$ is continuous, increasing and convex, since

$$f'(v) = \sum_{k=1}^{n} k x_k v^{k-1} > 0; f''(v) = \sum_{k=1}^{n} k(k-1) x_k v^{k-2} \geq 0.$$

Indeed, $f''(v) = 0$ when $n = 1$. Given these properties, it is easy to notice that function $f(v)$ crosses the line $y = P$ in only one point. To ensure that at such point $v < 1$, a necessary and sufficient condition is

$$f(1) > P \implies \sum_{x=1}^{n} x_k > P.$$

Figure 2.1 represents such a situation.

The above condition requires that the sum of the amounts to be received by the payer of P is greater than P itself.

2.3 Finding the IRR: Newton's method

Even when we know that a unique solution to equation (2.2) exists, it is not possible to solve the equation explicitly, unless we are in very special cases (degree of the polynomial lower than or equal to 4). However, it is always possible to apply numerical methods to find the roots of the polynomial equation. In this section, we review Newton's method, that belongs to the class of iterative methods for finding the roots of an equation. It is also called the method of tangents.

Fix an initial starting point v_0, and let us denote with v^* the solution of our equation. We seek for an algorithm that iteratively lowers the distance between a point at which the function is evaluated (the "current" point that the algorithm

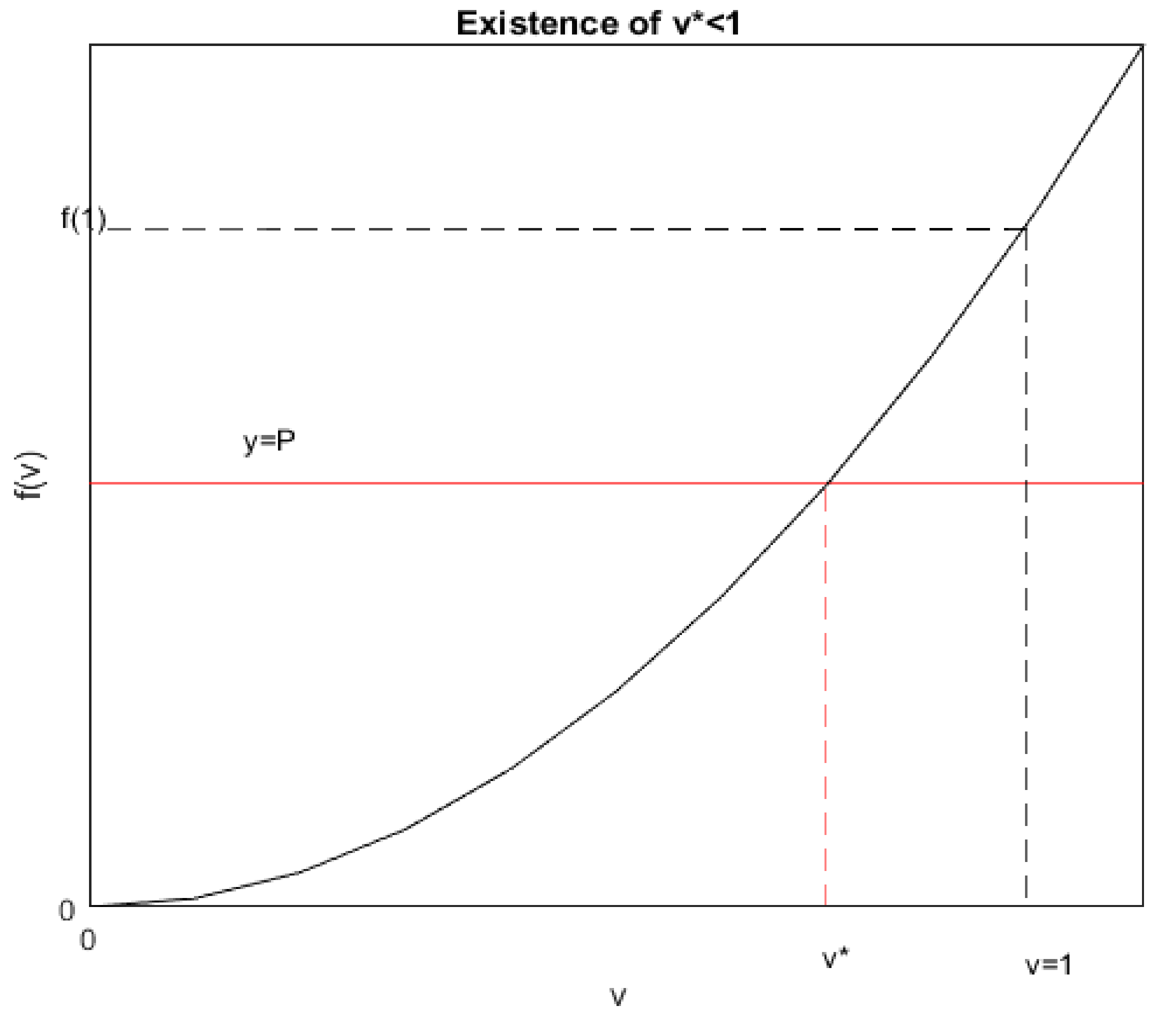

Figure 2.1: $f(1) > P$ is a necessary condition for existence of a unique positive solution v^*.

identifies) and the solution. In practice, the algorithm identifies a succession of points $v_0, ..., v_n$ that approximates the solution, i.e. such that

$$\lim_{n \to \infty} v_n = v^*.$$

In every iterative procedure, the succession is obtained specifying a "rule", i.e. a function F that allows to obtain the value of a point in the succession from the previous one, $v_{n+1} = F(v_n)$. Indeed, the distance between v_n and v^*, $|v_n - v^*|$ will be lower than the distance between v_{n-1} and v^*, $|v_{n-1} - v^*|$, i.e. will be decreasing with n.

Newton's method identifies the succession v_n following this reasoning. Start with v_0 greater than v^*. Due to convexity of $f(v)$, the tangent to f at v_0 will cross the function in one point, $v_1 \in (v^*, v_0)$ that will be closer to v^*. Iterating this procedure we approximate v^*, as can be appreciated from Figure 2.2.

The algorithm stops when the distance between two successive points is low enough, i.e. fixed an ϵ, when

$$|v_n - v_{n-1}| < \epsilon.$$

Indeed, v_n can be obtained in closed form at any n starting from $n-1$:

$$f'(v_n) = \frac{f(v_n) - P}{v_n - v_{n+1}},$$

$$v_{n+1} = v_n - \frac{f(v_n) - P}{f'(v_n)}.$$

and thus the algorithm will stop when $\frac{f(v_n)-P}{f'(v_n)} < \epsilon$ and return v_{n+1} as the approximate solution.

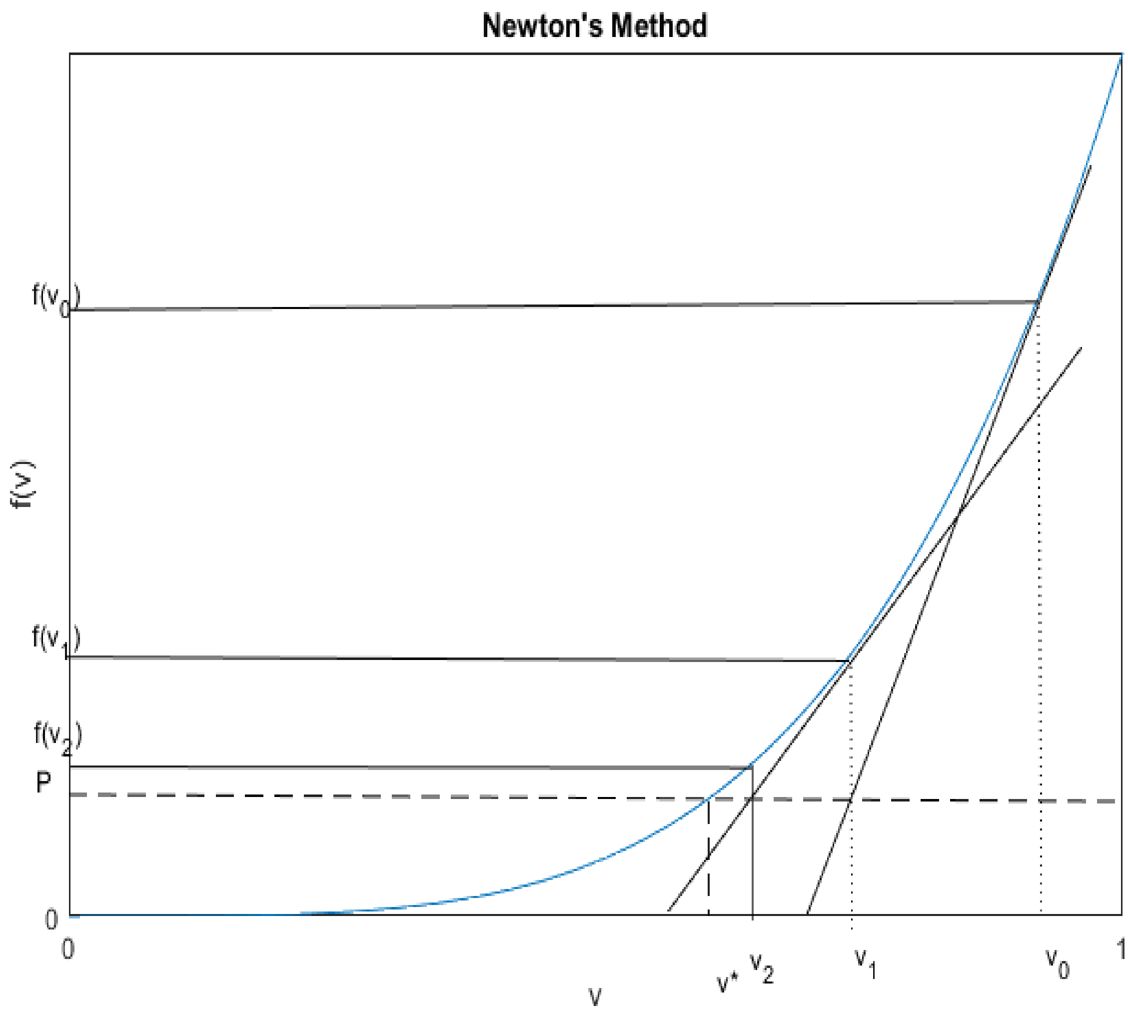

Figure 2.2: A graphical representation of Newton's method

2.4 Exercises

Exercise 2.1. Consider a contract $\mathbf{x} = \{50, 38, -100\}$, $\mathbf{t} = \{0, 1, 2\}$. Compute the IRR (at yearly basis). Determine the amount Δx_0 to be added to the first payment to obtain an IRR of 8%. (Answers:$i = 8.438\%; \Delta x_0 = 0.5487$ euros).

Exercise 2.2. Consider a financial market where the following assets are traded:

- an asset $\mathbf{x} = \{7, 2.5, 3, 87\}/\mathbf{t}$ priced 90 euros at time 0;
- an asset $\mathbf{y} = \{0, 4.5, 4, 32\}/\mathbf{t}$ priced 22 euros at time 0,

$\mathbf{t} = \{1, 2, 3, 4\}$ years. Compute the IRR of title $\mathbf{z}/\mathbf{t} = \mathbf{x}/\mathbf{t} + \mathbf{y}/\mathbf{t}$. What is the present value of $\mathbf{z}$ according to the exponential law whose annual compound interest rate i is the IRR? (Answers: $i = 6.25\%; W(0, \mathbf{z}) = 112.$).

Exercise 2.3. Calculate the semi-annual IRR of the financial transaction $\{-95, 4, 99\}$ / $\{1, 2, 3\}$, with time measured in years. Compute the amount x to be added to the last payment to obtain a semi-annual IRR of 6%. (Answers: $i = 2.084\%; x = 16.4409.$)

Exercise 2.4. Given the transaction $\{-55, 10, 50\}/\{0, 0.5, 1\}$, with time measured in years, compute its IRR i^*. By how much time Δt_2 must the last payment be postponed to obtain an IRR of 9%? (Answers: $i^* = 9.976\%; \Delta t_2 = 0.11435$ years.)

Exercise 2.5. A bank offers to finance a loan $S_0 = 100000$ euros, to be reimbursed after 4 years with an interest computed at an annual simple interest rate of 9.5%. An entrepreneur, who wants to enter the loan, objects that the interest rate is higher than the maximum interest rate prescribed by law, that is fixed to an IRR of 9% (annual). Compute the IRR i^* and check whether the entrepreneur is right. Compute the maximum interest I that the bank can require by law after 4 years. (Answers: No, he is not right, since the IRR i^* is 8.3851%. $I = 41158.161$ euros).

3 Annuities and amortization (redemption) of a loan

An *annuity* is a financial transaction involving periodic non-negative cash flows, named instalments. We denote the annuity as **a** and with R_k the k-th period instalment, $k = 1, ...n$, where n, the number of payments, is the annuity duration (term). An annuity whose payments go on forever ($n = +\infty$) is called a *perpetuity*. Annuities (and perpetuities) can be classified according to the moment at which the instalment is paid:

- in an *ordinary* annuity each payment is due at the end of the period;
- in an *annuity due* payments occur at the beginning of each period.

They are also classified according to their starting date:

- in an *immediate annuity* the beginning of the contract coincides with the date at which the agreement is signed;
- in a *deferred annuity* there is a delay between the date at which the contract is signed and the first periodic payment, called the deferral period.

3.1 Evaluating annuities with constant instalments under the exponential law

Let us consider an annuity **a**, with constant instalments equal to R. In the following, we calculate its value according to an exponential law with periodic discount factor v, where the reference period is the distance between two successive instalments.

3.1.1 Ordinary immediate annuity

Let us consider an ordinary immediate annuity of duration n, starting at $t_0 = 0$. The first payment will occurr thus at $t_1 = t_0 + 1 = 1$ and the last at $t_n = t_0 + n = n$. The present value of the annuity, $W(0, \mathbf{a})$ will be:

$$W(0,\mathbf{a}) = \sum_{k=1}^{n} x_k(1+i)^{-k} = \sum_{k=1}^{n} x_k v^k = R\sum_{k=1}^{n} v^k.$$

The value of the annuity is thus the sum of the first n elements of a geometric progression with first term equal to Rv and common ratio v. Hence, if $v \neq 1$,

$$\begin{aligned} W(0,\mathbf{a}) &= R\sum_{k=1}^{n} v^k = R\frac{1}{1-v}\sum_{k=1}^{n} v^k(1-v) = R\frac{1}{1-v}\sum_{k=1}^{n}(v^k - v^{k+1}) = \\ &= Rv\frac{1-v^n}{1-v} = R\frac{1-(1+i)^{-n}}{i} = Ra_{\overline{n}|i}. \end{aligned} \tag{3.1}$$

The symbol $a_{\overline{n}|i}$ reads "a figured n with rate i" and is thus the present value of an annuity with unit payments ($R = 1$) occurring at the end of each period.
When $v = 1$, i.e. when $i = 0$,

$$W(0,\mathbf{a}) = R\sum_{k=1}^{n} 1^k = Rn.$$

Indeed, $a_{\overline{n}|0\%} = n$.
Notice that, from formula (3.1), an interesting thing can be noticed. Rewriting the formula as

$$ia_{\overline{n}|i} + v^n = 1, \tag{3.2}$$

it follows that the sum of the present value of an annuity with n instalments of amount i and of the present value of one euro payable after n years is exactly 1. Indeed, it is fair to pay 1 at 0 in exchange for the interest rate i paid each year and the repayment of the loan 1 after n years. It is indeed equivalent to have the same amount at one date or after n years, if in the meanwhile the interests accrued are paid.

3.1.2 Perpetuity

Let us now consider the case of a perpetuity:

$$W(0, \mathbf{a}) = \sum_{k=1}^{+\infty} Rv^k = R \lim_{n \to +\infty} \sum_{k=1}^{n} v^k = R \lim_{n \to \infty} a_{\overline{n}|i} = Ra_{\overline{+\infty}|i} = R\frac{v}{1-v} = \frac{R}{i}.$$

Indeed, the last equalities hold true when $-1 < v < 1$. Otherwise, notice that if $v \leq -1$ then $\lim_{n \to \infty} v^n$ does not exist, and that if $v \geq 1$ it is equal to $+\infty$.

3.1.3 Temporary immediate annuity due

An annuity due is equivalent to an ordinary annuity with $n-1$ instalments, together with a first instalment at time t_0. Its present value, considering $t_0 = 0$ is:

$$W(0, \mathbf{a}) = R + Ra_{\overline{n-1}|i} = R(1 + a_{\overline{n-1}|i}) = R(1 + v\frac{1-v^{n-1}}{1-v}) = R\underbrace{\frac{1-v^n}{1-v}}_{\ddot{a}_{\overline{n}|i}}.$$

The symbol $\ddot{a}_{\overline{n}|i}$ should read as "a dots figured with n at interest rate i". Notice, first of all, that the following relation holds:

$$W(0, \mathbf{a}) = R\ddot{a}_{\overline{n}|i} = R\frac{1}{v}a_{\overline{n}|i}.$$

Notice that, since $d = 1 - v$, we have that

$$\ddot{a}_{\overline{n}|i} = \frac{1-(1-d)^n}{d}.$$

From the above relationship, it follows that the relation

$$d\ddot{a}_{\overline{n}|i} + v^n = 1$$

holds, with an interpretation similar to formula (3.2).

3.1.4 Perpetuity due

The value of a perpetuity due can be obtained easily from the value of an ordinary one:

$$W(0,\mathbf{a}) = \frac{R}{i}\frac{1}{v} = \frac{R}{d}.$$

Notice that, in terms of unitary annuities, $\ddot{a}_{\overline{\infty}|i} = a_{\overline{\infty}|i} + 1$.

3.1.5 Deferred annuities

When the annuity is deferred m periods, the first instalment is paid m periods after the time at which the contract is signed, t_0. Let us consider the contracts described above. If they are differed m periods, their time- m value is computed as above. For instance, an ordinary n-period annuity, with constant instalments R, deferred m periods, has time- m value equal to

$$W(m,\mathbf{a}) = Ra_{\overline{n}|i}.$$

Due to the decomposability property of the exponential law, the time-0 values of such annuities can be simply obtained by discounting the time-m value for the deferral period $[t_0, t_0 + m]$ and thus multiplying $W(m,\mathbf{a})$ by v^m. For instance, introducing the symbol ${}_m a_{\overline{n}|i}$ to identify the value of an n period unitary annuity deferred m periods, the time-0 value of the above-described ordinary deferred annuity is thus

$$W(0,\mathbf{a}) = R\ {}_m a_{\overline{n}|i} = Ra_{\overline{n}|i} v^m.$$

Similarly, the values of other annuity types are as follows:

- deferred, due temporary annuity: $W(0,\mathbf{a}) = R_m \ddot{a}_{\overline{n}|i} = R\ddot{a}_{\overline{n}|i} v^m$;
- deferred, ordinary perpetuity: $W(0,\mathbf{a}) = R\ {}_m a_{\overline{\infty}|i} = v^m R a_{\overline{\infty}|i} = \frac{R}{i} v^m$;
- deferred, perpetuity due: $W(0,\mathbf{a}) = R\ {}_m a_{\overline{\infty}|i} = v^m R \ddot{a}_{\overline{\infty}|i} = \frac{R}{i} v^{m-1}$

Notice that the following relation holds:

$${}_m a_{\overline{n}|i} = a_{\overline{n+m}|i} - a_{\overline{m}|i},$$

that formally states the equivalence between an m-period deferred annuity of duration n and an ordinary annuity of duration $n + m$ to which the value of the first m instalments is subtracted, i.e. without the first m instalments.

3.2 Amortization (redemption) of a loan

The amortization of a loan is a financial transaction that rules the exchange of an amount S at time 0 against an immediate annuity. Usually, the annuity is an ordinary and temporary one. Notable examples of a amortizations include the reimbursement of a personal loan or of a mortgage and the balance sheet pro quota imputation of the costs to buy an item.
The transactions we consider have the following profile: $\mathbf{x} = \{S, -R_1, -R_2, ..., -R_n\}$ / $\mathbf{t} = \{0, 1, 2, ..., n\}$, with $S > 0, -R_k \leq 0, k = 1, ...n$ where n is the duration of the annuity, i.e. the number of periods for which the instalments are paid. The transaction will be fair, according to a certain exponential law. Let us consider that precise law and let us consider the case of an ordinary immediate annuity from now on, to which we will refer as usual as to $\mathbf{a} = \{R_1, R_2, ..., R_m\}/\{1, 2, ..., n\}$. For every $0 \leq k \leq n$, fairness implies that the accumulated value of the transaction is equal to the opposite of its residual value:

$$M(k, \mathbf{x}) = -V(k, \mathbf{x}) = V(k, \mathbf{a}),$$

where the last equality holds because all the instalments of our considered annuity are non-positive. Indeed, considering the above relationship at time 0,

$$S = \sum_{k=1}^{n} R_k (1+i)^{-k},$$

which is also called the initial closure condition. The accumulated (or capitalized) value $M(k, \mathbf{x})$ is then equal to $V(k, \mathbf{a})$, that is the residual value of the instalments yet to be paid. Thus, we can define it as D_k, i.e. as the residual debt (value of the debt to be repaid) at time k. For $k \geq 1$, the recursive equation for the accumulated value derived in Section 1.11 holds:

$$M_k = (1+i)M_{k-1} + x_k,$$

which, in the context of an amortization reads

$$D_k = (1+i)D_{k-1} - R_k. \tag{3.3}$$

The above equation states that the residual debt value at time k is equal to the residual debt after having paid the last instalment, i.e. at the beginning of period k, appropriately capitalized, that is, increased by the interests matured during the k-th period, decreased by the instalment paid at the end of the k-th period, R_k to redeem the initial debt S. Equation (3.3) allows us to decompose each instalment R_k in two components:

$$R_k = \underbrace{iD_{k-1}}_{I_k\text{: Interest quota}} + \underbrace{D_{k-1} - D_k}_{C_k\text{: Amortization quota}}. \tag{3.4}$$

The interest quota I_k repays the additional insterest charged in the k-th period, which is computed as a fraction i of the residual debt at the beginning of the period. The amortization quota C_k is the part of the k-th instalment that contributes to decrease the residual debt. Such decomposition is not arbitrary, but is determined by the exponential law underlying the contract. Its practical relevance is associated to the different fiscal treatment that interests on debt have: the interest quota is usually fiscally deductible.

Notice that the following important relations hold:

$$\begin{aligned}
&D_0 = S \\
&D_n = 0 \\
&\sum_{k=1}^{n} C_k = \sum_{k=1}^{n} (D_{k-1} - D_k) = D_0 = S \\
&\sum_{l=k+1}^{n} C_l = \sum_{l=k+1}^{n} (D_{l-1} - D_l) = D_k.
\end{aligned}$$

The second equation above, $D_n = 0$, is called the final closure condition. Usually, the reimbursement of debt S is described through the so-called *amortization plan*, a table collecting the values of the relevant quantities D_k, R_k, I_k and C_k for $k = 0, ...n$.

3.3 Most common types of amortization

Fixed an initial sum S, the number n of instalments of the annuity and the interest rate i of the exponential law underlying the contract, different options are available to fix the instalments R_k of the annuity. The most common choices are two, defining the so-called French and Italian amortization types.

3.3.1 French amortization

In a French amortization, the instalments of the annuity are constant: for every k, $R_k = R$. Financial fairness at time 0 implies that the value of such constant instalments is determined as:

$$\begin{aligned} S &= V(0, \mathbf{a}) \\ S &= R a_{\overline{n}|i} = R\frac{1-v^n}{i} \\ R &= \frac{Si}{1-v^n} \qquad (3.5) \end{aligned}$$

Filling the amortization plan requires then the following steps:

1. determine the constant instalment R;
2. starting from $k = 1$, and $D_0 = S$, compute D_k using the recursive equation (3.3), and compute I_k and C_k, according to their definitions as in (3.4).

Indeed, for each k, alternatively, one can compute C_k first rather than D_k:

$$\begin{aligned} I_k &= iD_{k-1} \\ C_k &= R - I_k \\ D_k &= D_{k-1} - C_k. \end{aligned}$$

Let us now compute D_k, I_k, C_k at any k, starting from the value of R, computed according to equation (3.5):

$$
\begin{aligned}
D_k &= V(k,\mathbf{a}) = Ra_{\overline{n-k}|i} = R\frac{1-v^{n-k}}{i} = \frac{Si}{1-v^n}\frac{1-v^{n-k}}{i} = S\frac{1-v^{n-k}}{1-v^n};\\
C_k &= D_{k-1} - D_k = R\frac{1-v^{n-k+1}}{i} - R\frac{1-v^{n-k}}{i} = \frac{R}{i}(v^{n-k} - v^{n-k+1}) =\\
&= Rv^{n-k}\frac{1-v}{i} = Rv^{n-k+1};\\
I_k &= iD_{k-1} = iR\frac{1-v^{n-k+1}}{i} = R(1-v^{n-k+1}).
\end{aligned}
$$

Obviously, when filling the plan, the final residual debt value must be zero, i.e. the closure condition $D_n = 0$ must hold.

3.3.2 Italian amortization

In an Italian-type amortization, the amortization quotas are held constant throughout the amortization plan, i.e. $C_k = C$ for every $k = 1, ...n$. C can be obtained easily as

$$C = \frac{S}{n},$$

because the condition $S = \sum_{k=1}^{n} C_k = nC$ must hold. Starting from the first row, the amortization plan can be filled, using the recursive equations

$$
\begin{aligned}
I_k &= iD_{k-1}\\
R_k &= I_k + C\\
D_k &= D_{k-1} - C. \qquad (3.6)
\end{aligned}
$$

The relevant quantities at time k have then these expressions in an Italian amortization:

$$
\begin{aligned}
C &= \frac{S}{n} \\
D_k &= \sum_{h=k+1}^{n} C_k = (n-k)C = S\frac{n-k}{n} \\
I_k &= iD_{k-1} = i(n-k+1)C = iS\frac{n-k+1}{n} \\
R_k &= C + I_k = C + i(n-k+1)C = C(1+i(n-k+1) = S\frac{1+i(n-k+1)}{n}.
\end{aligned}
$$

3.3.3 Plans with non-constant fixed instalments or amortization quotas

It is possible to freely choose the profile of either the instalments or the amortization quotas and accordingly set the amortization plan. In choosing the profile of the amortization quotas, the constraint $\sum_{k=1}^{n} C_k = S$ must be fulfilled to ensure that the initial debt S is entirely reimbursed at the final date. The amortization plan is filled as in the Italian amortization case, where the starting point is the amortization quota, using the recursive relations between the relevant variables, once the C_ks are fixed. When the instalment profile is fixed, instead, the condition $S = \sum_{k=1}^{n} R_k(1+i)^{-k}$ must be met, and the amortization plan is compiled as in the French amortization case, in which the value of the instalment represents the starting point of the procedure, once the R_k's are fixed.

3.3.4 Plans with pre-amortization

In an amortization plan, an initial *pre-amortization* period can be present. This period represents a deferral of m periods in the repayment of debt. In such situation, only the interest quotas are paid for the first m periods, and then, starting from period $m+1$ the initial debt S is repaid as usual, through the payment of n amortization quotas, that can be determined according to one of the rules we have described in the previous sections. The amortization plan is constructed accordingly, noticing that $C_k = 0$ for $k \leq m$ and that the closure condition $D_{n+m} = 0$ holds at $n+m$. The limiting case is the reimbursement in a unique solution, that is indeed a plan with $n-1$ pre-amortization periods and where only one amortization quota is paid at time $t=n$.

3.4 Bonds and the bond markets

Together with loans, that we analyzed in the previous section, bonds constitute the most important form of investment/financing transactions. The bond is a financial instrument that allows the *issuer* to raise money from third parties (the market), promising a reimbursement in exchange. Indeed, issuing a bond, as well as taking on a loan, generates an initial inflows, followed by a series of outflows. Issuing a bond consists in one transaction from the point of view of the issuer, that is however split in several debt securities, that can be acquired by different parties. Issuing a bond that can be accessed by investors is a demanding process, both on the legal side (complying with current law) and on the placement side (finding the financiers).

Both firms and states issue bonds: securities issued by the former are called *corporate bonds*, by the latter are called *government bonds*. We refer to the placement process at issuance as to the *primary market* for bond securities. The placement can occur in various forms: at a fixed established price or through a public auction of some type. In the former case, which is the standard for corporate bonds, the process involves a consortium of financial intermediaries, that help structure the issuance and then place the security through a distribution channel. Government bonds, instead, are usually placed through auction procedures. In Italy, short-term bonds (BOTs) are placed via a competitive auction, medium and long-term ones via a marginal price auction.

After the placement, investors can trade (exchange) the securities in the *secondary market*. These markets are organized, in the sense that they need to comply with a certain series of rules that should grant transparency and efficiency in trades, and electronic. In Italy, there are two main markets where bonds and other debt securities can be traded: the MOT (Mercato Obbligazionario Telematico), that can be accessed by any investor and the MTS (Mercato all'ingrosso dei Titoli di Stato), where Government Bonds only are traded in big volumes by market specialists (intermediaries).

3.4.1 Standard bond types

Zero-coupon bonds

Zero-coupon bonds are securities that promise the investor the payment of a fixed sum C (the principal or notional, or face value) at a future date T (the maturity). To buy the security the investor pays a price P at time $t < T$. $T - t$ is called the time to maturity. The difference between C and P, $C - P$ is the interest. If t coincides with the time at which the security is issued, then P is the issue price, if t is after issuance, then P is the quote or buying price. A zero-coupon bond is described by the following couple of vectors of amounts and dates: $\mathbf{x} = \{-P, C\}$ and $\mathbf{t} = \{t, T\}$ and the value function that describes the transaction is such that $W(t) = P, W(T) = C$.

Example 3.1. In Italy, the BOTs (Buoni Ordinari del Tesoro) are the short-term bonds issued by the government. They are zero-coupon bonds. Their maturity (so far) has been 3 months, 6 months or 12 months. Bonds with different maturities (so-called flexible BOTs) have also been issued in the past. The last issuance of 3-month BOTs dates back to 2013. BOTs are placed through public auctions, that take place every month. CTZ (Certificati del Tesoro Zero-Coupon) are another zero-coupon bond security issued by Italian Government, whose maturity is longer than 12 months (24).
The last 6-month BOT (issued on March 29, 2017), for instance, has been issued at a price of 100.149 and had a time to maturity at issuance of 182 days. The difference between 100.149 and 100 gives the interest. j, the periodic interest rate, is then equal to $\frac{100-100.149}{100.149} = -0.1487\%$, that translates into an annual compound interest rate of -0.294%. The time measurement standard used in computation is usually ACT/360.

Coupon Bonds

Coupon bonds (or bullet bonds) are securities that promise the payment of n cash flows of a fixed amount I (the coupon) and the reimbursement of the notional capital $C > 0$ at maturity. I/C is called the coupon rate, while $I/C * m$, where m

is the number of coupons per year, is called the nominal annual interest rate. It is indeed, by convention, an annual simple interest rate, given the coupon rate. C is conventionally posed equal to 100 when quoting the bond. Denoting with Δt the interval between two successive coupons, and t the date of issuance, the obligation of the security, when the buying price P is equal to C, can be seen as an amortization plan with reimbursement through a unique payment, at time $t + n * \Delta t$. The financial transaction can be represented as $\mathbf{x}/\mathbf{t}$, where $\mathbf{x} = \{-P, I, I, I...., C + I\}$ and $\mathbf{t} = \{t, t + \Delta t, t + 2\Delta t, ...t + n\Delta t\}$. When $P = C$, the bond is said to be issued (or traded, if t corresponds to a date after the issuance) at par, it is above par if $P > C$ and below par if $P < C$.

Example 3.2. The Italian government regularly issues coupon bonds, called BTPs (Buoni del Tesoro Poliennali), having maturities ranging from 3 to 50 years (a bond maturing in 2067, that is usually referred to as the "Matusalem" bond, has recently been issued). The coupons are paid every 6 months. The last 10-year BTP has been issued with a coupon rate of 1.10%, and a nominal annual interest rate of 2.20% at a price of 99.63. The IRR of such a bond was 2.25%.

In the case in which $P = C$, the IRR can be obtained by solving the following equation:

$$-C + Ia_{\overline{n}|i} + Cv^n = 0$$
$$-C + I\frac{1 - v^n}{i} + Cv^n = 0$$
$$i = \frac{I}{C}.$$

Thus, the IRR expressed on a periodic basis is equal to the coupon rate. Notice however that the IRR on an annual basis will be different from the nominal annual interest rate.

Invoice price and trading price.

Consider a coupon bond, whose last coupon has been paid at time t_0. Consider now a transaction to exchange such bond occurring at a time t. t is $t_0 < t < t_1$, where t_1 is the time at which the next coupon will be paid to the bondholder. It is natural to consider the benefits from holding the bond, i.e. the payment of the coupon paid at t_1, as belonging partly to the seller at time t and partly to the buyer. Indeed, the amount I paid at t_1 will be splitted among the two parties, proportionally to the holding period. Indeed, the price the the buyer will need to pay will need to account for this:

$$P_{tq} = P + A,$$

where

$$A = I \frac{t - t_0}{t_1 - t_0},$$

P is the trading price and P_{tq} is called the *invoice* or *tel quel* price. Notice that A is the accrued interest between time t_0 and t when investing the initial capital C for a period $t - t_0$ at the simple annual interest $i = \frac{I}{C}m$. Using P as the trading price in the market, that is, a price that does not include the per quota value of the coupon, allows to better compare prices of obligations whose coupons are stripped at different dates.

Example 3.3. Consider a coupon bond with coupon $I = 3$ euros paid every 6 months, face value $C = 100$, acquired at $t = 0$ at the price $P = 96$ euros and delivering the following stream of payments: $\mathbf{x} = \{-96, 3, 3, 103\}$ at dates $\mathbf{t} = \{0, 1, 7, 13\}$ months. The bond has maturity 13 months, with coupon rate $i = 3\%$, and annual nominal rate 6%. The next coupon will be paid in 1 month, i.e. 5 months over the 6 month period from the last coupon payment have passed. The invoice price of the bond is thus:

$$P_{tq} = P + A = 96 + 3 * \frac{5}{6} = 96 + 2.5 = 98.5.$$

3.5 Exercises

Exercise 3.1. Consider a transaction that consists in buying, at $t = 0$ and at the price $P = 200$ euros, an immediate perpetuity with monthly instalment 1 euro. Compute its IRR i^*, expressed on an annual basis. Compute the value W

of the transaction after 10 years according to the exponential law identified by i^* and decompose it in its accumulated value M and residual value V. (Answers: $i^* = 6.1678\%, W = 0, V = 200; M = -200$).

Exercise 3.2. Consider an immediate perpetuity $\mathbf{a}_1$, with semi-annual instalments $R_1 = 10$ euros and an immediate temporary annuity due $\mathbf{a}_2$ with duration 13 semesters and instalments $R_2 = 100$ euros. Determine the net present value W of the portfolio $\mathbf{x} = \mathbf{a}_1 + \mathbf{a}_2$, according to an exponential law defined by an instantaneous interest rate $\delta = 0.05$ on an annual basis. Cosider buying $\mathbf{x}$ at time 0 at price W. Determine the IRR on an annual basis, i^* and the accumulated and residual values M and V, respectively, 6 months after the transaction, according to an exponential law identified by the annual interest rate i^*. (Answers: $W = 1518.843, i^* = 5.1271\%, M = -1344.76, V = 1344.76$).

Exercise 3.3. A financial institution buys an immediate annuity $\mathbf{a}'$, composed of 5 annual instalments $R' = 100$ euros and sells 2 annuities r'', deferred 2 years, and having 3 annual instalments $R'' = 50$ euros. Describe the vectors $\mathbf{x}$ and $\mathbf{t}$ defining the overall position of the institution. Assuming the cost paid at time 0 to set up the position is $P = 180$ euros, compute the IRR i^* of the transaction on an annual basis. Compute the accumulated and residual value at $t = 1$ of the transaction, including the payment of price P, according to the exponential law identified by i^*. (Answers: $i^* = 7.3212\%$, $M(1) = -93.1782, V(1) = 93.1782$.)

Exercise 3.4. Consider a perpetuity $\mathbf{a}_1$ and an annuity $\mathbf{a}_2$ with duration $m = 40$, both with quarterly instalments $R = 1200$ euros. Compute the present value of the portfolio $\mathbf{x} = \mathbf{a}_1 + \mathbf{a}_2$, using an exponential law with instantaneous interest rate intensity $\delta = 0.036$ on an annual basis. Answer the same question considering a perpetuity and an annuity due instead. In the case of the ordinary perpetuity and annuity, compute the residual portfolio value V at time $t = 4$ months. (Answers: $W(0, \mathbf{x}) = 172862.93$ in the ordinary case, $W(0, \mathbf{x}) = 174425.72$ in the “due" case. $V(4/12, \mathbf{x}) = 172547.57$).

Exercise 3.5. Consider a loan of S= 10 million euros to be reimbursed through an Italian amortization plan in 50 semi-annual instalments at an interest rate $i = 7\%$.

Determine R_{21}, decompose it in C_{21} and I_{21} and compute D_{21}, the residual debt after the payment of such instalment.
(Answers: $C = 200000; D_{21} = 5800000; I_{21} = 420000; R_{21} = 620000$)

Exercise 3.6. Bank Red proposes, for an initial debt value S euros, a non-standard amortization plan in 4 semi-annual instalments, with $i = 4\%$ and in which each instalment is exactly 4/5 of the amortization quota of the previous instalment. Determine S and fill the amortization plan, if the maximal instalment is 50 000 euros.

Exercise 3.7. A loan of S=10000 euros has to be reimbursed in 6 quarterly instalments, with $i = 3\%$, with the following characteristics:

1. the first two instalments are pre-amortization ones;
2. the third and fourth instalments each reimburse 1/4 of the initial debt;
3. the last two instalments are equal.

Fill the amortization plan.

Exercise 3.8. A firm takes on a mortgage of $S = 70000$ euros, to be reimbursed in 4 annual instalments. The redemption is carried out with the following characteristics:

- $i = 6\%$;
- the first three amortization quotas are equal;
- the last amortization quota is half the first.

Fill the amortization plan.

Exercise 3.9. Mr. Brown gets a loan from a bank for an amount S, to be reimbursed via a non-standard amortization plan in 3 quarterly instalments. The interest rate charged is $i = 4\%$, and each instalment is 90% the previous one. Compute the maximum loan S_{max} that Mr. Brown can get, if he knows he will not be able to reimburse more than 10000 euros each quarter and fill the corresponding amortization plan.

4 Additional exercises

Exercise 4.1. Mr. Smith asks Bank Green for a loan of 10000 euros, to be reimbursed after 1 year 3 months, with an interest of 840 euros. Compute the periodic interest rate j and the annual internal rate of return i^* of the transaction. Bank Blue is willing to give Mr. Smith the same amount, to be reimbursed after the same amount of time, charging an annual simple interest equal to 7.1 %. Compute the periodic interest rate j_B and the internal rate of return i^*_B of the transaction proposed by Bank Blue. Which proposal will Mr. Smith accept, and why?

Exercise 4.2. A bank proposes a loan of 15000 euros to an entrepreneur, to be reimbursed after T years with an interest of 5000 euros. Determine T, knowing that the annual compound interest rate charged is 4%. Another bank proposes to the entrepeneur a loan with same amount for T' years, charging the annual simple interest rate $i' = 5\%$. Determine T' so that the amount the entrepreneur needs to reimburse is the same proposed by the first bank. Tell to which bank will the entrepreneur ask for the loan. (Solution: $T = 7$ years, 4 months, $T' = 6$ years 8 months. He will get the loan from the first bank.)

Exercise 4.3. An individual wants to get a mortgage, for S euros, to be reimburse according to an amortization plan with 6 quarterly instalments, in which the first three are pre-amortization instalments, and the remaining three instalments are constant. The interest rate charged is 7% and each instalment can not be greater than 20000 euros. Compute the maximum capital that the individual can get, and fill in the corrisponding amortization plan. (Solution: $S = 57\,959.68$.)

Exercise 4.4. Mr. Red has to choose among two investment alternatives:

- Account Green, according to which the invested capital grows at a $i_G = 3\%$ annual compound rate;
- Account Yellow, according to which the invested capital grows at a $i_Y = 4\%$ annual simple interest rate.

Mr. Red wants to keep his money invested until the invested capital has increased by 25%. Determine first of all the time T_G needed to reach the goal in the case of

Account Green, the period interest rate $j_G(0, T_G)$. Compute the same quantities T_Y and $j_Y(0, T_Y)$ in the case of account Yellow. Determine in which of the two accounts Mr.Red will invest. (Solution: $j_G = 25\%$, $T_G = 7$ years, 6 months and 18 days, $j_Y = 25\%$, $T_Y = 6$ years and 3 months, he picks account Yellow. Day count convention: 30/360.)

Exercise 4.5. A firm agrees on a loan with a financial institution, to be reimbursed in 4 annual constant instalments. The amortization plan is constructed with the following characteristics:

- annual compound interest rate $i = 6\%$,
- each instalment cannot be greater than 100000 euros,
- amortization quotas are increasing:each one is 110% the preceding one.

Compute the maximum loan the firm can get at these conditions and fill in the amortization plan with $S = S_{max}$. Si calcoli l'importo massimo del finanziamento che l'azienda può ottenere a queste condizioni e si compili il piano di ammortamento con $S = S_{\max}$.
(Solution: $S = 348\,685.20$ $C_4 = 100\,000$, $C_3 = C_4/110\%$, $C_2 = C_3/110\%$, $C_1 = C_2/110\%$.)

Exercise 4.6. Mr. Addison wants to invest 18000 euros for 3 years 4 months. He has two alternatives: 1) an investment remunerated at 4% annual simple interest rate; 2) an investment remunerated at 3.1% annual compound interest rate. Compute the interest earned in the two cases and the amount of time required in both cases to earn an interest of 5000 euros. (Solution: $I_1 = 2400$ euros, $I_2 = 1928$ euros. $T_1 = 6.94444$ years, $T_2 = 8.029$ years.)

Exercise 4.7. An online bank proposes an investment opportunity, the *Green Bank Account* that charges a 6% annual simple interest rate on the first 10000 euros invested and 2.4% on the amount exceeding 10000 euros. Mr. Lee wants to invest 15000 euros for one year. Compute the interest I he will earn and the internal rate of return of the investment. Mr. Lee, after 4 months, has to cash in his account.

What is the interest I' earned in this case and what is the internal rate of return i'^*? (Solution: $I = 720$ euros, $i^* = 4.8\%$. $I' = 240$ euros, $i'^* = 4.8772\%$.)

Exercise 4.8. An entrepreneur asks for a loan of $S = 140000$ euros to two banks (bank A and bank B). Bank A offers a loan that is to be reimbursed in two semi-annual instalments of $R = 71000$ euros. Bank B offers to finance the loan at the annual simple interest rate $i_S = 5.5\%$, reimbursed in one payment after 1 year 2 months. Determine the internal rates of return of the two alternatives. Which bank will the entrepreneur choose and why?
(Solution: $i^*_A = 1.9108\%, i^*_B = 5.4754\%$.)

5 A more general theory of financial laws

In Section 1 we have learned what a financial law is, and explored some properties and the use of the most used financial laws, namely the exponential law and the linear law. However, these are not the only possible financial laws underlying contracts. Indeed, two parties can always agree upon their own financial law that defines the rules of intertemporal equivalence between the amounts exchanged within the contract. In this Section, we will go through a more general theory of financial laws, that will allow us to define spot and forward value functions, to better describe forward exchanges. The first question that naturally arises is what are the minimal requirements for a certain function $v(t,s)$, that describes the time t-value of a unit of cash payable at s, to be considered a financial law, when the valuation time t is also the time at which the exchange starts.

Definition 5.1. A *spot value function* is a function $v(t,s)$, with $t \leq s$, that determines the time-t value of a unitary sum payable at time s that satisfies the following properties:

1. $v(t,s) > 0$ for every $0 \leq t \leq s$;
2. $v(s,s) = 1$;
3. if $t \leq s < s'$, then $v(t,s) > v(t,s')$.

The first property implies that it is not possible to have the guarantee of obtaining a euro at a future date paying a negative or a null price; the second property simply tells us that getting 1 euro at a certain instant is exactly equivalent to get 1 euro at the same instant; the third property is a monotonicity property, guaranteeing that the cost to differ the reimbursement of a loan is strictly positive. The monotonicity property, together with the fact that $v(t,t) = 1$, imply that $v(t,s) < 1$ when $t < s$.

Example 5.1. It is evident that the exponential law, that can be expressed as $v(t,s) = e^{-\delta(s-t)}$ enjoys the three above properties.

Example 5.2. The function $v(t,s) = 1 - k(s-t)$ with $0 < k < \frac{1}{s-t}$ is a spot value function. This function describes what is normally referred to as to the "commercial (or bank) discount" law.

5.1 The value function of a forward contract

It is important to generalize the description of financial laws by allowing the starting date of the contract (i.e. the time at which the exchange starts) to differ from the valuation time. Let us consider a transaction starting at a date T and ending at date $s \geq T$, but whose contracting date is $t \leq T$. The *forward value function* $v(t, T, s)$ defines the time-T value of 1 euro payable at s, as agreed upon at time t. Analogously to the properties of the *spot* value function we defined above, the function $v(t, T, s)$ satisfies the following properties:

1. $v(t, T, s) > 0$ for every $t \leq T \leq s$;
2. $v(t, s, s) = 1$, for $t \leq s$;
3. $v(t, t, s) = v(t, s)$, for $t \leq s$;
4. $v(t, T, s) > v(t, T, s')$ for $t \leq T \leq s < s'$;
5. $v(t, T, s) < v(t, T', s)$ for $T < T'$.

The first two properties have the same interpretation of the analogous properties for the spot value function. The third property states that the spot value function can be obtained as a particular case of the forward one, where t and T coincide. The last two properties require the monotonicity of the value function relative to the ending date of the investment horizon (the more distant, the lower the forward value function) and the starting date of the transaction (the more distant, the greater the value function). Indeed, the two last properties require that a lender, according to the financial law identified by $v(t, T, s)$ requires a compensation both when the reimbursement is postponed and when the loan start is moved backwards. In what follows, we will require two additional properties: the independence of the value function from the amounts and the coherence property between spot and forward transactions identified by a value function.

5.1.1 Independence from the amounts

We denote with $V(t; x_s)$ the time-t value of an amount x_s payable at $s \geq t$. The independence assumption implies that

$$V(t; x_s) = x_s v(t, s).$$

The value function does not depend from the amount that is exchanged: $v(t, s)$ can thus be interpreted as the discount factor over period $[t, s]$ or as the exchange factor between x_s at time s and $V(t; x_s)$ at time t. Accordingly, it is natural to identify the reciprocal of $v(t, s)$ with the accumulation factor for period $[t, s]$:

$$m(t, s) = \frac{1}{v(t, s)}.$$

5.1.2 Coherence property, decomposability and value additivity

The independence property stated above, that leads to the intepretation of $v(t, s)$ as a discount factor, induces us to assume also financial coherence between spot and forward contracts. It is indeed natural to think that the time-t value of a euro payable at s should coincide with the value $v(t, T, s)$ discounted from time T up to time t. This reasoning translates into the coherence property, that states that, for every $t \leq T \leq s$, the following relation holds:

$$v(t, s) = v(t, T, s)v(t, T).$$

In other terms:

$$v(t, T, s) = \frac{v(t, s)}{v(t, T)}.$$

The above expression clarifies that the coherence assumption implies that the forward value function is completely determined by the spot one, as the two terms on the r.h.s. are indeed spot value functions. In other words, it is sufficient to identify a spot value function and to assume the coherence property to appropriately identify a forward value function. Notice that, due to the independence assumption:

$$V(t,T;x_s) = \frac{V(t;x_s)}{v(t,T)} = x_s\frac{v(t,s)}{v(t,T)}.$$

We recall now that a financial law is decomposable if $v(t,s) = v(t,T)v(T,s)$, for $t \le T \le s$. Notice that, although the above expression is similar to the one implied by the coherence property, their financial meaning is totally different. The coherence property introduces a relationship between values of contracts signed at the same date t; the decomposability property determines the value of a transaction at a future date T, starting from the values of the transactions occurring at date t. Only when $v(t,T,s) = v(T,s)$, i.e. the value of the forward contract signed at t and starting at T coincides with the value of a spot contract starting at T the two expressions coincide. Also, the time-uniformity property, i.e. $v(t+\Delta t, s+\Delta t) = v(t,s)$ will not be assumed to hold.

Example 5.3. The commercial (bank) discount law,

$$v(t,s) = 1 - k(s-t)$$

is evidently time uniform, but it is not decomposable, since

$$v(t,T)v(T,s) = (1-k(T-t))(1-k(s-T)) \neq (1-k(s-t)) = v(t,s).$$

Example 5.4. Consider the financial law

$$v(t,s) = e^{-\frac{1}{2}(s^2-t^2)}, 0 \le t \le s.$$

This law is not time uniform, since

$$v(t+\Delta t, s+\Delta t) = e^{-\frac{1}{2}((s+\Delta t)^2-(t+\Delta t)^2)} \neq e^{-\frac{1}{2}((s)^2-(t)^2)} = v(t,s).$$

This law is decomposable, since

$$v(t,T)v(T,s) = \frac{e^{-\frac{1}{2}(T^2-t^2)}}{e^{-\frac{1}{2}(s^2-T^2)}} = e^{-\frac{1}{2}(s^2-t^2)} = v(t,s).$$

Example 5.5. The financial law

$$v(t,s) = e^{-\delta(t)(s-t)},$$

with $\delta(t)$ a time-dependent function is not decomposable. Indeed:

$$v(t,T,s) = \frac{v(t,s)}{v(t,T)} = \frac{e^{-\delta(t)(s-t)}}{e^{-\delta(t)(T-t)}} = e^{-\delta(t)(s-T)},$$

while

$$v(T,s) = e^{-\delta(T)(s-T)}.$$

In the following, while we will assume coherence between spot and forward value functions, we will not assume decomposability, that will be enjoyed only by some of the financial laws. Value additivity will instead be assumed. Indeed, considering a financial transaction $\mathbf{x} = \{x_1, x_2, ..., x_m\}/\mathbf{t} = \{t_1, t_2, ..., t_m\}$, the time t-value of the transaction enjoys the value additivity property if the time t-value can be obtained as the sum of the discounted values of the single amounts:

$$V(t;\mathbf{x}) = \sum_{k=1}^{m} V(t;x_k) = \sum_{k=1}^{m} x_k v(t,t_k).$$

5.2 Extension of the fundamental quantities to the general case.

All of the fundamental quantities we introduced in Section 1 can be extended to the more general setting introduced in this section. The periodic forward interest rate, the accumulation factor and the average rate of return can be defined as:

$$j(t,T,s) = \frac{1}{v(t,T,s)} - 1;$$

$$m(t,T,s) = 1 + j(t,T,s);$$

$$\gamma(t,T,s) = \frac{j(t,T,s)}{s-T}.$$

The notion of equivalent interest rates can also be extended to such more general case. Define now for simplicity the unitary compound interest rate, i.e. the annual compound interest rate:

$$i(t,T,s) = [1 + j(t,T,s)]^{\frac{1}{s-T}} - 1 = \left[\frac{1}{v(t,T,s)}\right]^{\frac{1}{s-T}} - 1.$$

Let us now also extend our definition of the instantaneous interest rate (or force of interest). For every $t \leq T$, we define the instantaneous interest rate as the limit of $\gamma(t,T,T+\Delta T)$ when ΔT, the length of the duration of the exchange, tends to 0:

$$\delta(t,T) = \lim_{\Delta T \to 0} \frac{j(t,T,T+\Delta T)}{\Delta T}.$$

We recover the definition of $\delta(t)$ gave in Section 1 if $T = t$. Notice that:

$$\delta(t,T) = \lim_{\Delta T \to 0} \frac{j(t,T,T+\Delta T)}{\Delta T} = \lim_{\Delta T \to 0} \frac{\frac{1}{v(t,T,T+\Delta T)} - 1}{\Delta T} = \lim_{\Delta T \to 0} \frac{\frac{v(t,T)}{v(t,T+\Delta T)} - 1}{\Delta T} =$$

$$= \frac{1}{m(t,T)} \lim_{\Delta T \to 0} \frac{m(t,T+\Delta T) - m(t,T)}{\Delta T} = \frac{1}{m(t,T)} \frac{\partial m(t,T)}{\partial T},$$

assuming that the limit exists finite. Hence:

$$\delta(t,T) = \frac{\partial \log m(t,T)}{\partial T} = -\frac{\partial \log v(t,T)}{\partial T}.$$

Example 5.6. Let us compute the instantaneous intensity of some of the laws we encountered so far:

- exponential law: $v(t,s) = e^{-\delta(s-t)}$;

$$\delta(t,s) = -\frac{\partial log v(t,s)}{\partial s} = -\frac{\partial log[e^{-\delta(s-t)}]}{\partial s} = \frac{\partial \delta(s-t)}{\partial s} = \delta.$$

- linear law: $m(t,s) = 1 + (s-t)i_s$;

$$\delta(t,s) = \frac{\partial log m(t,s)}{\partial s} = \frac{i_s}{1 + (s-t)i_s}.$$

- commercial discount law: $v(t,s) = 1 - k(s-t)$;

$$\delta(t,s) = -\frac{\partial log v(t,s)}{\partial s} = -\frac{\partial log[1-k(s-t)]}{\partial s} = \frac{k}{1-k(s-t)}.$$

- "exponential" law with time-dependent force of interest: $v(t,s) = e^{-\delta(t)(s-t)}$;

$$\delta(t,s) = -\frac{\partial log v(t,s)}{\partial s} = -\frac{\partial[-\delta(t)(s-t)]}{\partial s} = \delta(t).$$

Definition 5.2. We define the yield to maturity as the instantaneous interest rate h of the exponential law equivalent to $v(t,s)$ over the horizon $[t,s]$ of the transaction:

$$h(t,s) = \log\left[1 + i(t,s)\right] = -\frac{1}{s-t}\log v(t,s).$$

The above definition naturally extends to the case of forward contracts, being

$$h(t,T,s) = -\frac{1}{s-T}\log v(t,T,s).$$

It is worth recalling here that the only law for which $h(t,s) = h(t,T,s) = \delta(t,s) = \delta \in \mathbb{R}_+$ and for which $i(t,s)$ is constant is the exponential law.
Let us now derive the important relation existing between $\delta(t,s)$ and $m(t,s)$. Integrating over the interval $[t,s]$, we obtain:

$$\int_t^s \delta(t,u)du = \int_t^s \frac{\partial \log m(t,u)}{\partial u} du = \log m(t,s) - \underbrace{\log m(t,t)}_{=\log 1 = 0} = \log m(t,s).$$

Hence:

$$m(t,s) = e^{\int_t^s \delta(t,u)du};$$

$$v(t,s) = e^{-\int_t^s \delta(t,u)du}.$$

Also, it follows that the yield to maturity is the time average of the instantaneous interest rate intensity, fixed at time t over the interval $[T,s]$:

$$h(t,T,s) = \frac{1}{s-T}\int_T^s \delta(t,u)du.$$

The two properties of time uniformity and decomposability we have defined for the discount factors, can be alternatively stated as properties of the instantaneous interest rate. For a time uniform law, for every t and s,

$$\delta(t, s) = \delta(t + \Delta t, s + \Delta t).$$

As a consequence, $\delta(T, s) = \delta(t, s - \Delta t)$. For a decomposable law, we recall that

$$v(T, s) = \frac{v(t, s)}{v(t, T)} = v(t, T, s).$$

It follows that the relationship

$$e^{-\int_T^s \delta(T,u)du} = e^{-\int_T^s \delta(t,u)du},$$

must hold for every $s \geq T$, which implies that

$$\delta(t, s) = \delta(T, s).$$

In turn, this implies that the instantaneous interest rate will be independent from t:

$$\delta(t, s) = \delta(s), t \leq s.$$

This last relation constitutes a necessary and sufficient condition for decomposability. Summarizing, it follows that a law is decomposable if the instantaneous interest rate does not depend on the first variable over which it is defined, but depends on the second one only. Moreover, let us derive a general form of the discount factor $v(t, s)$ which guarantees decomposability. Starting from the relation between v and δ we have that

$$v(t, s) = e^{-\int_t^s \delta(u)du} = e^{-[F(s)-F(t)]},$$

with $F(u)$ being the primitive of $\delta(u)$. Defining $G(u) = e^{-F(u)}$, it follows that, for $t \leq s$,

$$v(t, s) = \frac{G(t)}{G(s)}.$$

This last equation implies that for any decomposable financial law, the discount factor can be expressed as a function of an arbitrary positive-valued increasing function of time t.

Remark 5.1. We recall here the result we already obtained in Section 1: the exponential law is the only financial law that is time uniform and decomposable. This can be proved easily, given that the exponential law is the only law exhibiting a constant instantaneous interest rate.

5.3 Exercises

Exercise 5.1. Consider a market where, at the valuation instant $t = 0$, the following instantaneous interest rate function applies: $\delta(0, s) = 2\% + 1\%s$. Determine the price of:

1. a ZCB paying 100 at time $s_1 = 1$ year;
2. an annuity, paying a rate $R = 10$ yearly and lasting 2 years;
3. a forward contract starting at $s_1 = 1$ year, paying 500 in $s_3 = 3$ years;
4. a forward contract paying 20 in $s_2 = 2$ years and 20 in $s_3 = 3$ years, starting at $s_1 = 1$ year. (Answers: 97.53; 19.18;461.56;37.77)

Exercise 5.2. Knowing that yields to maturity in a market are described by the function $h(0, t) = 3\% - 1\%e^{-t}$, compute $v(0, 2), v(0, 1, 2), i(0, 1, 2), h(0, 2)$ and $\delta(0, 2)$. (Answers: $v(0, 2) = 0.9443, v(0, 1, 2) = 0.9695, i(0, 1, 2) = 3.1457\%, h(0, 2) = 2.8647\%$ and $\delta(0, 2) = 3.1353\%$.)

Exercise 5.3. Consider a market in which, at $t = 0$, the following term structure holds, defined by the intensity:

$$\delta(0, s) = \begin{cases} 0.04 \text{ years}^{-1} & s \leq 1 \text{ year} \\ 0.035 + 0.005s \text{ years}^{-1} & s > 1 \text{ year.} \end{cases}$$

Determine the term structure of spot and forward interest rates (contracted at $t = 0$) for maturities equal to 0.5,1 and 1.5.

Exercise 5.4. Consider a market in which the following term structure of instantaneous interest rates holds (t expressed in years):

$$\delta(0,t) = 4\% \cdot e^{-0.1t}.$$

Compute, in such market:

- The price P_1 at $t_0 = 0$ of an annuity with $n = 3$ constant semi-annual instalments of amount $R = 200$ euros;
- the price P_2 at $t_0 = 0$ of a forward contract payable at $t_1 = 1$, providing a cash flow $C = 400$ euros at $t_2 = 1.5$.

(Answers: $P_1 = 577.8276; P_2 = 393.0012$.)

Exercise 5.5. Consider a market in which the yield to maturities of the contracts, $h(0,t)$ are given by the equation:

$$h(0,t) = \frac{3.5\%}{1+t},$$

where t is expressed in years. Compute:

- the price P of an annuity with semi-annual constant instalments equal to $R = 2000$ euros and duration $n = 3$;
- the price P' of a forward contract signed at $t = 0$, starting at $t = 0.5$ years, paying 5000 euros at t=5 years;
- the instantaneous interest rate $\delta(0,1)$.

(Answers: $P = 5900.5447, P' = 4913.2612, \delta(0,t) = 3.5\% \frac{1}{(1+t)^2}$.)

Exercise 5.6. Consider a market in which, at time $t = 0$, the following term structure of instantaneous interest rates holds:

$$\delta(0,s) = \begin{cases} 0.05 & s \leq 0.5 \\ 0.04 + 0.02 & 0.5 < s \leq 1 \\ 0.06 & s > 1. \end{cases}$$

Determine the term structure of yields to maturity, spot and forward, for the dates $\{0, 0.5, 1\}$. Determine the time-$t = 0$ price of an annuity due with semi-annual constant instalments $R = 510$ euros and duration $n = 2$. (Answer: $h(0, 0.5) = h(0, 0, 0.5) = 0.05, h(0, 1) = 0.0525, h(0, 1.5) = 0.05550844, h(0, 0.5, 1) = 0.055,$ $h(0, 1, 1.5) = 0.06152; P = 981.3238.$)

6 Market Prices and Absence of Arbitrage

In the previous sections we almost always started from the description of the financial law underlying a contract to determine its price, according to the principle of financial fairness. In this section, we will face the problem of trying to recover the "laws" underlying a market, observing the prices that are formed after the trades. In particular, we will consider a (risk-free) bond market under some assumptions that will allows us to determine such financial law.

6.1 Perfect market hypotheses

Consider a market where assets with deterministic payoffs at fixed maturities are traded continuously, i.e. at any instant. We assume that in this market some ideal conditions hold:

1. Absence of frictions:
    - no transaction costs: buying prices (bid) and selling prices (ask) coincide;
    - no taxes;
    - any quantity of assets can be traded (i.e. no minimum quantity for a trade);
    - short sales are allowed: it is possible to sell assets that are not held by the agent. Indeed, this implies that the agent can always become debtor in a transaction. These last two hypothesis together allow an agent to trade any quantity $Q \in \mathbb{R}$ of an asset, even fractionary or negative (a negative amount is interpreted as a short sale).
2. Perfect Competition:
    - agents are profit maximizers: they always prefer more to less, this is captured by the impatience postulate;
    - agents are price-takers: none of them can influence the price of the asset.

3. Absence of default risk: promised obligations are always fulfilled.

4. Finally, we assume that in the market no arbitrages exist.

6.2 No arbitrage property

Definition 6.1. An arbitrage is a financial transaction $\mathbf{x}/\mathbf{t}$, with $\mathbf{x} = \{x_0, x_1, ..., x_n\}$, $\mathbf{t} = \{t_0, t_1, ..., t_n\}$ such that either

- $x_0 \geq 0$ and $x_k \geq 0$ for every k, with at least one k s.t. $x_k > 0$ (arbitrage of the first kind);

- or $x_0 > 0$ and for every k $x_k \geq 0$ (arbitrage of the second kind).

An arbitrage is thus a financial transaction that guarantees a certain stream of surely non-negative cash flows, with at least one strictly positive cash flow. Interpreting x_0 as the cost paid in t_0 to get the future cash flows $x_1, ...x_n$, $c = -x_0$, we can interpret the two types of arbitrages this way. In an arbitrage of the first kind an agent obtains at least one positive future cash flow, while paying either 0 or a negative sum (i.e. receiving some money) at time 0. In an arbitrage of the second kind, an agent pays a negative sum at 0 (i.e. he receives money) to get a series of non-negative cash flows. In any of the two alternatives, the agent obtains a certain profit, through a financial transaction built at $t = 0$. With an arbitrage of the second kind, he makes an immediate sure profit (x_0), followed possibly by other positive inflows. In an arbitrage of the first kind, instead, he may not make an immediate profit (if $x_0 = 0$), but he will surely obtain at least one positive cash flow at a future date. It is evident that any arbitrage opportunity, in a perfect market with the characteristics described above, will provide a way of generating an infinite amount of money for the agent setting up the arbitrage.

6.3 The law of one price

The most immediate consequence of the absence of arbitrages assumption is the so-called law of one price.

Theorem 6.1. (**The law of one price**) Assume a perfect market with no arbitrages. Then, the time-t price of an asset paying a series of certain cash-flows is unique.

Proof. Suppose without loss of generality, that two zero-coupon bonds A and B exist in the market, both paying 1 at time T, having prices p_A and p_B, with $p_A > p_B$. Then, an agent can simultaneously sell A and buy B at time t, obtaining $p_A - p_B > 0$ at time t. At time T, the agent will obtain 1 because he bought bond B, and will need to reimburse 1 because he sold A, thus having a null cash flow. Thus, an arbitrage of the second kind exists, which contradicts our hypothesis. □

The theorem says that if two assets promising the same future cash flows are traded in a perfect market where no arbitrages exist, they need to have the same price.

6.4 No arbitrage and zero-coupon bonds

In the previous section, we have defined as $v(t, s)$ the time-t price of 1 euro payable at s. Indeed, in a bond market, the price of a zero-coupon bond with unitary notional, to which we refer to as to $p(t, s)$, can be interpreted as a "market-based" version of $v(t, s)$, defining, under the assumptions we have made for our market, the "financial law" underlying the market. Indeed, the absence of arbitrage implies that $p(t, s)$ must be positive for any t, s, $p(t, s) > 0$, with $t \leq s$ and that its price at maturity, $p(s, s)$ will be equal to 1.

Remark 6.1. The no arbitrage assumption does not prevent $p(t, s)$ to be greater than 1. In order to guarantee that this holds true, we have assumed impatience.

Impatience allows us to state the following theorem as well.

Theorem 6.2. (**Prices are decreasing in maturities**) Consider two zero-coupon bonds with maturities s' and s'', with $s'' > s' \geq t$ traded in our perfect market. If no arbitrage holds, then the zero-coupon bond prices are decreasing in their maturities: $p(t, s') > p(t, s'')$.

Proof. Suppose instead $p(t, s') \leq p(t, s'')$. It is then possible to construct an arbitrage by:

a) buying at t the ZCB with maturity s';

b) going short (i.e. selling) at t the ZCB with maturity s'';

c) buying at s' the ZCB with maturity s''.

The investor than has the following cash flows:

Date	t	s'	s''
a)	$-p(t,s')$	1	0
b)	$p(t,s'')$	0	-1
c)	0	$-p(s',s'')$	1
Total	$p(t,s'')-p(t,s')$	$1-p(s',s'')$	0

The strategy returns the investors a series of non-negative cash flows: $p(t,s'') - p(t,s') \geq 0$ by assumption and $1-p(s',s'')$ even if unknown at time t is positive due to the impatience postulate. Hence, no arbitrage does not hold, which contradicts the assumption of the theorem. □

So far, we have considered unitary notional zero-coupon bonds. Consider now a market in which, together with unitary zero coupon bonds, also ZCB having notional x_s that will be reimbursed at maturity s are traded. Denote with $P(t;x_s)$ their time-t price. Then, the following theorem, concerning the independence of prices from amounts, holds.

Theorem 6.3. (**Independence from amounts**) Assume perfect markets and no arbitrage opportunities. Then, the following equality holds: $P(t;x_s) = x_s p(t,s)$.

Proof. Assume instead $P(t;x_s) < x_s p(t,s)$. Then, the following arbitrage opportunity exists:

a) buy the zcb with notional x_s at time t;

b) sell x_s unit notional zero-coupon bonds with maturity s.

The payoffs of such a strategy are

Strategy \ Date	t	s
a)	$-P(t;x_s)$	x_s
b)	$x_s p(t,s)$	$-x_s$
Total	$x_s p(t,s) - P(t;x_s)$	0

The strategy is an arbitrage opportunity, because the payoff of the strategy at time t is stictly positive, while the payoff at maturity s is 0. A symmetric strategy can be constructed when the reversed inequality holds. Then, the no arbitrage assumption is violated and the theorem is proved. □

Consider now that in the market, a series of zero-coupon with different maturities is traded. Then, it is possible to construct assets with more elaborate structures as portfolios of such zero-coupon bonds. Consider an asset with the following payoff, $\mathbf{x} = \{x_1, x_2, ..., x_n\}$ at maturities $\mathbf{t} = \{t_1, t_2, ..., t_n\}$. Denote with $P(t;\mathbf{x})$ the time t-price of such an asset. Then, the following theorem holds.

Theorem 6.4. (**Linearity of prices**) Consider a perfect market with no arbitrage opportunities. Then, the following equality holds:

$$P(t;\mathbf{x}) = \sum_{k=1}^{n} x_k p(t,t_k).$$

Proof. Assume, on the contrary that, for instance, $P(t;\mathbf{x}) < \sum_{k=1}^{n} x_k p(t,t_k)$. Then, the following strategy:

a) buy the asset with payoff $\mathbf{x}/\mathbf{t}$ at time t;

b) sell at time t an amount x_k of zero coupon bonds with maturity t_k for $k = 1, ...n$,

is an aribtrage. Indeed:

Date	t	t_1	t_2	...	t_n
a)	$-P(t; \mathbf{x})$	x_1	x_2	...	x_n
b_1)	$x_1 p(t, t_1)$	$-x_1$	0	...	0
b_2)	$x_2 p(t, t_2)$	0	$-x_2$	...	0
....	...	...	....	...	...
b_n)	$x_n p(t, t_n)$	0	0	...	$-x_n$
Total	$\sum_k x_k p(t, t_k) - P(t; x_s)$	0	0	0	0

This violates our assumption of no arbitrage. An analogous result obtains with the reversed inequality $P(t; \mathbf{x}) > \sum_{k=1}^{n} x_k p(t, t_k)$, because one can set up a symmetric arbitrage strategy. □

6.5 Forward contracts and no arbitrage: implied forward rates.

Let us now consider the implications of no arbitrage on the price of forward contracts. Since we have a bond market in mind, unitary forward contracts, whose value at time t will be denoted as $p(t, T, s)$, akin to the $v(t, T, s)$ we have defined in the previous sections, are contracts signed at t that rule the exchange of a unitary zero-coupon bond with a certain maturity s at a predetermined date T, with $t \leq T \leq s$.
No arbitrage implies the following theorem.

Theorem 6.5. (**Implied forward prices**) If no arbitrage opportunities exist in the "perfect" market, then the following equality holds:

$$p(t, T, s) = \frac{p(t, s)}{p(t, T)}, t \leq T \leq s.$$

Proof. Suppose $p(t, s) > p(t, T)p(t, T, s)$ instead. Then, the following strategy:

a) take a short position on a ZCB with maturity s;

b) buy spot $p(t, T, s)$ units of the ZCB with maturity T;

c) buy a forward contract to obtain at T 1 ZCB with maturity s, is an arbitrage. Consider indeed the cash flows delivered by such strategy:

Date	t	T	s
a)	$p(t,s)$	0	-1
b)	$-p(t,T,s)p(t,T)$	$p(t,T,s)$	0
c)	0	$-p(t,T,s)$	1
Total	$p(t,s) - p(t,T,s)p(t,T)$	0	0

All cash flows are null, a part from the one at time t, that is $p(t,s) - p(t,T,s)p(t,T) > 0$ by assumption, hence contradicting the no arbitrage assumption. □

7 The term structure of interest rates

In this section, we will consider the perfect market we outlined in the previous section, where no arbitrage is assumed to hold, and learn how to recover the set of interest rates for each maturity (term structure), starting from the prices of the obligations observed in such market. The idea is that market conditions that rule the exchanges for the different terms (maturities) can be inferred, given the prices observed at time t. Throughout this section, we will consider the term structure obtained from market prices for zero-coupon bonds with maturities $t_k = t + k, k = 1, ..., m$.

7.1 Spot term structure

Consider first having the knowledge, at time t, of the spot prices $P(t, x_k)$ of the zero coupon maturing at time t_k and paying a notional x_k. In the perfect, arbitrage-free market we are considering, we know that the independence property holds, according to which

$$P(t, x_k) = x_k p(t, t_k) \implies p(t, t_k) = \frac{P(t, x_k)}{x_k}.$$

The set $\{p(t, t_k)\}$, with $k = 1, ...m$ represents the *term structure of spot discount factors*, as identified by the prices of spot transactions that can be performed in the market. Analogously, we define the *term structure of interest rates* as the set of spot interest rates with maturity t_k that can be obtained from spot prices:

$$i(t, t_k) = \left[\frac{1}{p(t, t_k)}\right]^{\frac{1}{t_k - t}} - 1.$$

As defined in Section 6, these interest rates are expressed on the basis of one unit of measure of time (i.e. for us from now on, on an annual basis). Finally, the market prices define also the term structure of yields to maturity, that can be obtained as:

$$h(t,t_k) = \log[1 + i(t,t_k)] = \log\left[\left[\frac{1}{p(t,t_k)}\right]^{\frac{1}{t_k - t}}\right] = -\frac{1}{\underbrace{t_k - t}_{k}}\log p(t,t_k).$$

Suppose instead we observe the forward prices of zero-coupon bonds with maturities t_k for a notional x_k instead of the spot ones. Suppose for instance we have the prices of all the forward transactions in which the delivery date (the date at which the contract starts) is one period prior to maturity: $P(t,t_{k-1},x_k)$, for $k = 1,...m$. Again, for the independence property, it follows that

$$P(t,t_{k-1},x_k) = \frac{P(t,x_k)}{p(t,t_{k-1})} = x_k\frac{p(t,t_k)}{p(t,t_{k-1})} = x_k p(t,t_{k-1},t_k).$$

Hence the set of prices $p(t,t_{k-1},t_k)$, with $k = 1,...m$,

$$p(t,t_{k-1},t_k) = \frac{P(t,t_{k-1},x_k)}{x_k}$$

defines the term structure of forward discount factors. Analogously:

$$i(t,t_{k-1},t_k) = \left[\frac{1}{p(t,t_{k-1},t_k)}\right]^{\frac{1}{t_k - t_{k-1}}} - 1,$$

$$h(t,t_{k-1},t_k) = \log[1 + i(t,t_{k-1},t_k)] = -\frac{1}{t_k - t_{k-1}}\log p(t,t_{k-1},t_k),$$

define the term structures of forward interest rates and yields to maturity, respectively.

7.2 Implied rates and term structure

So far, we retrieved the term structure of spot rates from spot prices and the term structure of forward rates from the prices of forward contracts. However, it is always possible, in an arbitrage-free, perfect market, to obtain the term structure of forward interest rates from the prices of spot transactions and viceversa, making use of the Theorem of implied prices, that states the link existing in such market between forward prices and spot prices. Indeed, the following relation holds, that links the price of forward contracts to the spot ones:

$$p(t,T,s) = \frac{p(t,s)}{p(t,T)},$$

from which we can get the forward interest rates as a function of the spot prices:

$$i(t,T,s) = \left[\frac{1}{p(t,T,s)}\right]^{\frac{1}{s-T}} - 1 = \left[\frac{p(t,T)}{p(t,s)}\right]^{\frac{1}{s-T}} - 1.$$

Recalling that

$$1 + i(t,s) = \left[\frac{1}{p(t,s)}\right]^{\frac{1}{s-t}} \implies p(t,s) = [1 + i(t,s)]^{-(s-t)},$$

the above expression reads

$$1 + i(t,T,s) = \left[\frac{[1+i(t,T)]^{-(T-t)}}{\underbrace{[1+i(t,s)]^{-(s-t)}}_{[1+i(t,s)]^{-(s-T+T-t)}}}\right]^{\frac{1}{s-T}} = [1+i(t,s)]^{\frac{s-T}{s-T}} \left[\frac{1+i(t,s)}{1+i(t,T)}\right]^{\frac{T-t}{s-T}}.$$

Indeed, it follows that

$$1 + i(t,T,s) = [1 + i(t,s)]\, f,$$

where f is a factor that takes value:

- > 1 if $i(t,s) > i(t,T)$, i.e. if the term structure of interest rates is increasing between s and T. In this case the forward rate between T and s agreed upon at t is greater than the spot rate with maturity s;

- < 1 if $i(t,s) < i(t,T)$, i.e. if the term structure of interest rates decreases between s and T. In this case the forward rate between T and s agreed upon at t is lower than the spot rate with maturity s;

- equal to one if $i(t,s) = i(t,T)$. In this case, the spot rate $i(t,s)$ equals the forward rate $i(t,T,s)$.

Remark 7.1. A similar relationship holds for yields to maturity, for which

$$h(t,T,s) = h(t,s) + \frac{T-t}{s-T}\left[h(t,s) - h(t,T)\right],$$

with analogous considerations relating the spot yields to maturity and the forward ones.

Analogously, it is possible to compute the spot interest rate term structure from the forward one. Indeed, it is easy to notice that, for every k

$$p(t_0,t_k) = p(t_0,t_0,t_1)p(t,t_1,t_2)p(t,t_2,t_3)...p(t,t_{k-1},t_k) = \prod_{l=1}^{k} p(t,t_{k-1},t_k).$$

Hence,

$$(1+i(t_0,t_k))^{-(t_k-t_0)} = \prod_{l=1}^{k}(1+i(t_0,t_{l-1},t_l))^{-(t_l-t_{l-1})},$$

which implies that

$$1+i(t_0,t_k) = \prod_{l=1}^{k}(1+i(t_0,t_{l-1},t_l))^{\frac{t_l-t_{l-1}}{t_k-t_0}}.$$

Hence, it follows that 1+ the spot rate equals the geometric weighted average of 1+ the forward rates, where the weights are given by the lengths of the periods over which the forward rates are considered.

Remark 7.2. The spot yields to maturity can be recovered instead as an arithmetic weighted mean of the forward ones:

$$h(t_0,t_k) = \sum_{l=1}^{m}\frac{t_l-t_{l-1}}{t_k-t_0}h(t,t_{l-1},t_l).$$

Remark 7.3. In this section, we have learned how to recover the term structure for the maturities for which zero-coupon bonds (or forward contracts on zero-coupon bonds) are present. This implies considering a discrete set of maturities at which prices, interest rates and yields to maturity can be computed. First, the reader should be aware that it is possible, similarly, to define a term structure starting from the prices of coupon bonds, using their link to zero-coupon bond prices identified by Theorem 7.4. in the previous section. Second, in general, the term structure of interest rates can be computed for the continuous set of maturities between t and t_k, even for those maturities for which no traded contracts are present. This can be done by assuming an underlying function that interpolates the points that lie between the ones for which the spot and forward interest rates can be computed along the lines we traced above. This process allows to describe the so-called *interest rate curve* and the *yield curve*, which are curves defined on a continuous set of maturities.

Remark 7.4. Consider a bond paying cash flows $\mathbf{x} = \{x_1, x_2, ...x_n\}$ at dates $\mathbf{t} = \{t_1, t_2, ...t_n\}$. Its price P, fair at time t_0 can be obtained using the interest rate term structure:

$$P = \sum_{k=1}^{n} x_k(1 + i(t_0, t_k))^{-(t_k - t_0)}.$$

Its internal rate of return, by definition, will be the rate i^* such that:

$$P = \sum_{k=1}^{n} x_k(1 + i^*)^{-(t_k - t_0)}.$$

Hence, i^* can be obtained as a functional average of the spot interest rates, weighted according to the cash flows and payment dates of the bond.

7.3 Exercises

Exercise 7.1. In a market, at time t, the following quotes (for 100 euros notional) are observed:

- the spot price of a ZCB with maturity 2 years is 95;
- the spot price of a coupon bond with maturity 3 years and paying an annual 2% coupon is 98;
- the forward price (delivery date 1 year) of a ZCB with maturity 2 years is 97.

Determine the term structure of spot and forward interest rates implied by no arbitrage.
Determine also the forward price (delivery date 1 year from now) of a ZCB with maturity 3 years. (Answers: $i(0,2) = 0.025978; i(0,1) = 0.021053, i(0,3) = 0.027086$, forward rates follow. $p(0,1,3) = 94.2384$);

Exercise 7.2. Consider a market in which the term structure of yields to maturity is

$$h(0,t) = \begin{cases} 4\% & t \leq 1.5; \\ 5\% & t > 1.5 \end{cases}$$

In such market, compute the price P of a coupon bond with maturity 3 years, paying an annual coupon at a rate of 10%, and having notional $C = 4000$ euros. Compute also the forward price, delivery date 1 year, of a ZCB with maturity 3 years and notional $C' = C$. (Answers: $P = 4533.3658; P_{ZCB}(0,1,3) = 3583.3365$).

Exercise 7.3. Consider a market in which a ZCB with maturity 6 months and notional 1000 has a spot price of 980 euros and a coupon bond with semi-annual coupons (nominal annual interest rate 4%) and maturity 1 year traded at par are quoted. In such market, determine the price P of an annuity with 2 semi-annual instalments of an amount 4000 euros. Knowing that in the same market, for more distant maturities, greater than one year, the following term structure of interest rate holds:

$$i(0,t) = 2\% + 1\% * t, t > 1,$$

determine the forward interest rates $i(0,0.5,1), i(0,1,1.5), i(0,1.5,2)$. (Answers: $P_a = 7764.7059; i(0,0.5,1) = 0.01958; i(0,1,1.5) = 0.02430; i(0,1.5,2) = 0.05514$).

Exercise 7.4. Consider a market in which the following term structure of discount factors hold:

$$p(0,t) = \frac{2}{1+e^{at}},$$

with time expressed in years and $a = 0.1$. Compute the spot interest rates and yields to maturity for maturities $T = 0.25, 0.5, 0.75$. (Answers: $i(0,0.25) =$

$0.0516; i(0, 0.5) = 0.05193; i(0, 0.75) = 0.05226$, $h(0, 0.25) = 0.0050312; h(0, 0.5) = 0.050625; h(0, 0.75) = 0.050937$).

Exercise 7.5. Consider a market in which, at time $t = 0$, the following assets, paying cash flows $\mathbf{x}$ and $\mathbf{y}$ at $\mathbf{t} = \{1, 2, 3\}$, respectively, with

- $\mathbf{x} = \{10, 10, 10\}$ euros, traded at 27 euros;
- $\mathbf{y} = \{0, 10, 10\}$ euros, traded at 17.5 euros.

Suppose that the asset paying cash flows $\mathbf{z} = \{20, 70, 70\}$ at the same dates is traded at the spot price 140 euros. Construct an arbitrage involving the two assets in the market and $\mathbf{z}$, obtaining an immediate profit of 15 euros, with null cash flows at later dates.

Exercise 7.6. Consider a market in which at time $t = 0$, are quoted:

- a ZCB with notional 100 euros and maturity 2 years, with spot price 87.5;
- a ZCB with notional 200 euros and maturity 3 years, traded at a forward price (delivery date 1 year from now) of 178 euros;
- a ZCB with notional 100 euros and maturity 3 years, traded at a forward price (delivery date 2 years from now) of 95 euros.

Determine the term structure of spot and forward interest rates implied by these prices under no arbitrage. Then, assuming that an annuity, paying 3 annual instalments $R = 100$ euros, is traded at 270 euros, tell if it is possible to build an arbitrage. (Answers: $i(0, 1) = 7.05\%, i(0, 2) = 6.9\%, i(0, 3) = 6.35\%, P = 264.06; Yes$).

Exercise 7.7. Consider a market in which at time $t = 0$ the following term structure of instantaneous intensities holds:

$$\delta(0, s) = 5\%(1 + s).$$

In such market, compute the price P of an annuity with $n = 3$ annual constant instalments $R = 500$ euros and the forward price P' of a ZCB with maturity 3 years and notional $C = 4000$ euros, starting at time 2.

Exercise 7.8. Consider a market in which at time $t = 0$, the following assets are quoted:

- A spot ZCB, with maturity $T_1 = 0.5$ years, notional 100 euros and price 95;
- a forward ZCB, with maturity $T_2 = 1$ year, notional 100 euros and price 99 euros to pay at T_1.

Determine the term structure of spot interest rates. In the same market, an annuity with semi-annual instalment $R = 1000$ euros and $n = 2$ is quoted at the price $P = 1250$ euros. Assuming that it is not possible to short more than 10000 euros of notional of the first ZCB, set up an arbitrage of the second kind guaranteeing a gain G at $t = 0$. Determine G.

8 Interest-rate risk and duration

So far, we have not introduced any uncertainty relative to the value of financial instruments. In this section, while abstracting from any possible uncertainty relative to the payoffs of the instruments we consider, we will introduce the issue of the uncertainty regarding the term structure of interest rates, that we will treat as an exogenous variable. This issue is of practical relevance. Consider a transaction to buy a zero-coupon bond with maturity s at time t. Such transaction will be ruled in our perfect, arbitrage-free market by the conditions that can be expressed by the term structure of interest rates identified by the time-t prices of bonds. At a future date T the "rules" of the market will be identified by the prices that will emerge at that date, which need not be identical to the ones observed at time t. Hence, an agent who buys at time t a zero-coupon bond with maturity s knows that he will get the notional with certainty if he holds the bond until maturity, but he does not know what the "rules" of the market will be at a future date T, $t < T < s$ at which he may want to sell his bond. This section will address such problem, which is indeed an *interest rate risk* issue.

8.1 Interest rate risk: an example

Consider an agent who buys a unitary zero-coupon bond at time t with maturity s years for its price $p(t,s) = 1 * (1 + i(t,s))^{-(s-t)}$. He knows the value of this bond will be exactly 1 at maturity, s. However, he does not know, in principle, its value at time $T, t < T < s$. Indeed, this value is equal to $p(T,s) = 1 * (1 + i(T,s))^{-(s-T)}$, that is a function of $i(T,s)$, which will define the market conditions at the future date T. Suppose that a sudden change in the term structure of the interest rate occurs, right after the transaction, that transforms $i(t,s)$ into $i(t,s) + \Delta i$. Accordingly, the market price of the bond just bought will move and become: $p'(t,s) = (1 + i(t,s) + \Delta i)^{-(s-t)}$. The risk that such a change in the term structure happens is indeed the *interest rate risk*. The price variation, $p'(t,s) - p(t,s)$ will indeed be a function of Δi and $s - t$. Intuitively, the higher Δi, the higher the price change following a change in the term structure.

Example 8.1. Consider the following example. An agent buys a unitary zero-coupon bond at time t, with maturity T and $i(t,T) = 1\%$. Consider the effects of an increase $\Delta i = 0.5\%$ on $i(t,T)$: $i(t,T) = \Delta i + i(t,T)$ when $T = 1, 3, 5$ years.

Maturity	$P(t)$	$P'(t)$	$\Delta P = P'(t) - P(t)$
$T = 1$	0.9901	0.9852	-0.0049
$T = 3$	0.9706	0.9563	-0.0143
$T = 5$	0.9515	0.9283	-0.0232

8.2 Duration

It is evident from the above example that the price change following an interest rate change is higher (in both absolute and relative terms, since P(t)'s are decreasing with maturity) the more distant the maturity. This observation leads us to interpret the distance of a ZCB from maturity as a measure of its interest-rate risk, because the longer the time to maturity, the longer the time horizon for which the agent is exposed to the risk of variations in the term structure of interest rates. Let us now try to extend such measure to compute the risk borne by any bond, identified by its vectors of payments and dates $\mathbf{x}/\mathbf{t}$, defined as usual, $\mathbf{x} = \{x_1, x_2, ...x_n\}$ and $\mathbf{t} = \{t_1, t_2, ...t_n\}$. Let us first consider the risk associated to each x_k, $k = 1, ..., n$ as if it was the payoff of a zero coupon bond with maturity t_k. Our measure of such risk is equal to $t_k - t$, where t is the valuation instant. Thus, the risk associated to $\mathbf{x}/\mathbf{t}$ can be seen as composed by the aggregation of the risks associated to each of these payoffs. How to aggregate such risks? An appropriate choice consists in considering a weighted average of the individual risks, where the weights are chosen to account for the fact that each payoff contributes differently to the price of the bond. Indeed, the weight assigned to each payoff x_k will be

$$\pi_k = \frac{x_k p(t,t_k)}{P(t,\mathbf{x})} = \frac{P(t,x_k)}{P(t,\mathbf{x})},$$

the ratio of the time t-price of the zero-coupon bond with maturity t_k and notional x_k and the price of the bond, i.e. the percentage contribution (in terms of value) of the payoff of the bond having maturity t_k to the price of the bond.

Summarizing, we define the measure as

$$D(t,\mathbf{x}) = \sum_{k=1}^{n} \pi_k (t_k - t).$$

Such measure is called the (Macaulay) Duration of the bond, which is called also average financial duration. Its unit of measure is indeed the unit of measure of time.
It is usually defined on $\mathbf{x} \geq 0$, but in case $\mathbf{x}$ contains both positive and negative x_k's, the positive and negative parts are decomposed and two separate durations are computed. Expressing the zero-coupon bond prices $p(t,t_k)$ in terms of the corresponding interest rates we get

$$D(t,\mathbf{x}) = \frac{1}{P(t,\mathbf{x})} \underbrace{\sum_{k=1}^{n} x_k (1+i(t,t_k))^{-(t_k-t)}}_{\text{Dollar Duration (DD)}}.$$

Notice that, while the duration is not linear, Dollar duration is:

$$DD(t, \alpha\mathbf{x} + \beta\mathbf{y}) = \alpha DD(t,\mathbf{x}) + \beta DD(t,\mathbf{y}).$$

8.3 Duration of a portfolio

Assume that an investor holds more than one bond and wants to compute the duration of its whole portfolio, as a synthetic measure of the exposure of its whole financial position to interest rate risk. Consider for simplicity he holds two bonds, with payoffs $\mathbf{x}$ and $\mathbf{y}$ respectively. The duration of a portfolio of two such bonds, $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$ is

$$\begin{aligned} D(t,\mathbf{z}) &= \frac{DD(t,\mathbf{z})}{P(t,\mathbf{z})} = \frac{\alpha DD(t,\mathbf{x}) + \beta DD(t,\mathbf{y})}{P(t,\mathbf{z})} = \frac{\alpha D(t,\mathbf{x})P(t,\mathbf{x}) + \beta D(t,\mathbf{y})P(t,\mathbf{y})}{P(t,\mathbf{z})} = \\ &= \frac{\alpha P(t,\mathbf{x})}{P(t,\mathbf{z})} D(t,\mathbf{x}) + \frac{\beta P(t,\mathbf{y}}{P(t,\mathbf{z})} D(t,\mathbf{y}) = \pi_x D(t,\mathbf{x}) + \pi_y D(t,\mathbf{y}). \end{aligned}$$

Hence, the duration of a portfolio of bonds is the weighted arithmetic average of the duration of its components, with weights equal to the percentage invested in each bond. The above formula extends easily to the case of n assets. Suppose that the investor holds a percentage π_i of each of the n assets $x_1, x_2, ..., x_n$. Hence:

$$D(t, \mathbf{z}) = \sum_i \pi_i D(t, \mathbf{x}_i).$$

A typical problem related to duration is the one of an asset manager who has a certain duration target D^* and a budget constraint P^*, i.e. a certain wealth to invest in a set of assets. He has to solve the following system of linear equations, to find the portfolio composition that allows him to simultaneously meet the duration target and satisfy the budget constraint:

$$\begin{cases} \sum_i \alpha_i P(t; \mathbf{x}_i) = P^* \\ \sum_i \alpha_i DD(t, \mathbf{x}_i) = P^* D^*. \end{cases}$$

This system is a linear system of 2 equations in n unknowns.

8.4 Second order moment and variance

Duration is an arithmetic average. It represents the average distance to the payments of the asset, weighted by their percentage contribution to the price of the asset. It measures than the distance from the valuation instant of the barycentre of the temporal distribution of the weights π_k. Such distribution has thus a second order moment:

$$D^{(2)} = \sum_{k=1}^{n} \pi_k (t_k - t)^2,$$

and a variance

$$Var(\pi_k) = D^{(2)} - D^2.$$

This variance is a measure of the dispersion in time of the cash flows associated to the bond and it is equal to zero only in the case of zero coupon bonds.

8.5 Flat term structure

A notable particular case, whose implications we will examine in this subsection, is the case of a flat term structure of interest rates, where:

$$i(t,s) = i, t \leq s.$$

This case is particularly useful if we can compute the internal rate of return i^* of an asset, which as we have seen, can be interpreted as a functional average of the interest rates prevailing in the market. In such case, the price of an asset paying cash flows $\mathbf{x}$, equals

$$P(t,\mathbf{x}) = \sum_{k=1}^{n} x_k(1+i)^{-(t_k-t)} = P(i) = \sum_{k=1}^{n} x_k e^{-\delta(t_k-t)} = P(\delta.)$$

The duration, written in terms of δ, equals:

$$D(t,\mathbf{x}) = \frac{\sum_{k=1}^{n} x_k e^{-\delta(t_k-t)}(t_k-t)}{\sum_{k=1}^{n} x_k(1+i)^{-(t_k-t)}}.$$

Let us compute the derivative of the price w.r.t. δ:

$$P'(\delta) = -\sum_{k=1}^{n} x_k(t_k-t)e^{-\delta(t_k-t)} = -DD(t,\mathbf{x}) < 0;$$
$$P''(\delta) = \sum_{k=1}^{n} x_k(t_k-t)^2 e^{-\delta(t_k-t)} > 0.$$

This second derivative is equal to the numerator of the second-order moment $D^{(2)}$, since $D^{(2)}$ equals

$$D^{(2)}(t,\mathbf{x}) = \frac{\sum_{k=1}^{n} x_k e^{-\delta(t_k-t)}(t_k-t)^2}{\sum_{k=1}^{n} x_k(1+i)^{-(t_k-t)}}$$

Indeed, it follows that

$$D(t,\mathbf{x}) = -\frac{P'(\delta)}{P(\delta)}.$$

The duration thus equals the semi-elasticity of price w.r.t. δ, with sign changed.

The second order moment, $D^{(2)}(t, \mathbf{x})$ is equal to:

$$D^{(2)}(t, \mathbf{x}) = \frac{P''(\delta)}{P(\delta)},$$

Since it is a function of the second derivative of price w.r.t. δ, the second order moment is usually referred to as to the convexity term. Denoting with δ_0 the current value of the yield to maturity in a flat term structure, the effects of a sufficiently small variation $\Delta\delta$ in the interest rate of a flat term structure on the price of an asset can be approximated via Taylor's formula:

$$P(\delta_0 + \Delta\delta) = P(\delta_0) + P'(\delta_0)\Delta\delta + \frac{1}{2}P''(\delta_0)\Delta\delta^2 + o(\Delta\delta^2).$$

$$P(\delta_0 + \Delta\delta) = P(\delta_0) + -D(t, \mathbf{x})P(\delta)\Delta\delta + \frac{1}{2}D^{(2)}(t, \mathbf{x})P(\delta)\Delta\delta^2 + o(\Delta\delta^2).$$

Hence,

$$\frac{P(\delta_0 + \Delta\delta) - P(\delta_0)}{P(\delta)} \approx -D(t, \mathbf{x})\Delta\delta + \frac{1}{2}D^{(2)}(t, \mathbf{x})\Delta\delta^2.$$

If the Taylor expansion is stopped at the first order, the percentage price variation is approximately equal to the product of the change in the constant yield to maturity and the duration, changed of sign. If the approximation is stopped at the second order instead, an additional term, always positive, is added to the percentage price variation. If instead of performing the computations in terms of δ we reason in terms of the interest rate i, we get similar formulas:

$$P'(i) = -\sum_{k=1}^{n} x_k(t_k - t)(1 + i)^{-(t_k - t) - 1} = \frac{1}{1 + i}DD(t, \mathbf{x}) < 0.$$

$$\begin{aligned} P''(i) &= \sum_{k=1}^{n} x_k(t_k - t)(t_k - t + 1)(1 + i)^{-(t_k - t) - 2} = \\ &= \frac{1}{(1 + i)^2}\left[\sum_{k=1}^{n} x_k(t_k - t)^2(1 + i)^{-(t_k - t)} + \sum_{k=1}^{n} x_k(t_k - t)(1 + i)^{-(t_k - t)}.\right]. \end{aligned}$$

Approximating using Taylor's expansion, we obtain that:

$$\begin{aligned}\frac{\Delta P}{P(i)} &\approx \frac{P'(i)}{P(i)}\Delta i + \frac{1}{2}\frac{P''(i)}{P(i)}\Delta i^2 = \\ &= -\frac{1}{1+i}D(t,\mathbf{x})\Delta i + \frac{1}{2}\frac{1}{(1+i)^2}\left[D^{(2)}(t,\mathbf{x}) + D(t,\mathbf{x})\right]\Delta i^2.\end{aligned}$$

The quantity $-\frac{1}{1+i}D(t,\mathbf{x})$ is called modified duration.

8.5.1 Duration of an annuity

Consider an annuity $\mathbf{r}$ paying n regular instalments (one each period) equal to R. The price of the annuity is $P(0,\mathbf{r}) = Ra_{\overline{n}|i}$. Its dollar duration is equal to

$$DD(0,\mathbf{r}) = \sum_{k=1}^{n} kRv^k = Rd_{\overline{n}|i} = R\sum_{k=1}^{n} kv^k,$$

where $d_{\overline{n}|i}$ is equal to:

$$\begin{aligned}d_{\overline{n}|i} &= \frac{1}{1-v}\sum_{k=1}^{n}\left[kv^k - kv^{k+1}\right] = \\ &= \frac{1}{1-v}\left[v - v^2 + 2v^2 - 2v^3 + 3v^3 - ...\right] = \frac{1}{1-v}\left[v + v^2 + v^3 + ... + v^n - nv^{n+1}\right] = \\ &= \frac{1}{1-v}\left[a_{\overline{n}|i} - nv^{n+1}\right].\end{aligned}$$

Duration is thus

$$\begin{aligned}D(0,\mathbf{r}) &= \frac{DD(0,\mathbf{r})}{P(0,\mathbf{r})} = \frac{Rd_{\overline{n}|i}}{Ra_{\overline{n}|i}} = \frac{\frac{1}{1-v}\left[a_{\overline{n}|i} - nv^{n+1}\right]}{a_{\overline{n}|i}} = \\ &= \frac{1}{1-v}\left[1 - \frac{nv^{n+1}}{a_{\overline{n}|i}}\right] = \frac{1}{1-v} - \frac{nv^n}{1-v^n} = \frac{1+i}{i} - \frac{n}{(1+i)^n - 1}.\end{aligned}$$

8.5.2 Duration of a coupon bond

Let us compute now the duration of a coupon bond, paying n coupons I at a regular intervals and reimbursing C at maturity, after n periods. Indeed, the coupon bond can be seen as the sum of an annuity, $\mathbf{r}$ composed by the coupon payments, and a

ZCB $\mathbf{z}$ with maturity n. Hence, the dollar duration of the coupon bond $\mathbf{x}$ is

$$DD(0,\mathbf{x}) = DD(0,\mathbf{r}) + DD(0,\mathbf{z}) = Id_{\overline{n}|i} + Cnv^n.$$

The duration is then:

$$D(0,\mathbf{x}) = \frac{DD(0,\mathbf{x})}{P(0,\mathbf{x})} = \frac{Id_{\overline{n}|i} + Cnv^n}{Ia_{\overline{n}|i} + Cv^n} = \frac{\frac{I}{C}d_{\overline{n}|i} + nv^n}{\frac{I}{C}a_{\overline{n}|i} + v^n},$$

which depends on the coupon rate $\frac{I}{C}$, rather than on the coupon and the notional separately.

8.6 Exercises

Exercise 8.1. A financial institution holds a bond portfolio, composed by:

- 500 ZCBs with maturity 3 months;
- 200 ZCBs with maturity 6 months;
- 399 ZCBs with maturity 1 year.

Determine the duration D of this portfolio, if the term structure is flat and $i = 5\%$. The institution decides to increase the duration of its portfolio by 0.1 years, modifying its positions on the 6-month and 3-month bonds, increasing the former by α units and decreasing the latter by the same amount of units. Determine α. (Answers: $D = 0.562215, \alpha = 443.6527$).

Exercise 8.2. Consider, at time $t = 0$, a bond market with two traded contracts:

- bond $\mathbf{x}$, traded at 100 euros and having duration 5 years;
- the ZCB $\mathbf{y}$, maturity 1 year, traded at 300 euros.

Consider an investor with capital 1000 euros. Determine the minimum and maximum duration he can achieve for a portfolio of the two assets without short sales. Determine the portfolio (α_x, α_y) that he needs to hold if he wants to set his duration to 4 years. (Answers: $D_{min} = 1, D_{max} = 5$ years, $\alpha_x = \frac{30}{4}, \alpha_y = \frac{10}{12}$.)

Exercise 8.3. Consider a market in which, at time $t = 0$, the term structure of discount factors is described by the following function:

$$p(0, s) = 1 - ks,$$

with $k = 0.03$. Compute:

- The price P_1 and the duration D_1 of a coupon bond with maturity 2 years and annual coupon paid at a nominal annual interest rate 6%, notional 100 euros;
- The forward price P_2 payable at $T = 1$ year and 3 months, agreed upon at $t = 0$, to obtain 50 euros at $s =$2 years and 4 months.
- The instantaneous interest rate at 4 years. (Answers: $P_1 = 105.46, D_1 = 1.944813, P_2 = 48.31169, \delta(0, 4) = 0.034091$).

Exercise 8.4. A portfolio, whose total value is V=2 million euros, is composed for one fourth of its value by a BOT with maturity 3 months, and for the remaining part by an asset with duration $D_2 = 7$ years. Compute the duration of the portfolio. Suppose to substitute the second asset with a coupon bond $D_3 = 10$ years. Compute the overall value V' of the portfolio obtained under the hypothesis that its duration does not change relative to the initial portfolio. (Answers: $D = 5.3125; V' = 1.040$ mln euros).

Exercise 8.5. A public university has a debt $C = 200$ million euros outstanding with maturity $T = 3$ years at time $t = 0$. A monthly stream of perpetual payments of 55000 euros adds up to this debt. Compute the value and duration of the debt, assuming a flat term structure of interest rates with $i = 3.12\%$. Assuming again a flat term structure, compute the interest rate i' that would make the duration of the debt equal to 3 years. (Answers: $V = 221.45; D = 5.87$ years, $i' = 40.22\%$).

Exercise 8.6. Suppose the Italian public debt is entirely composed by 10-year BTPs with nominal annual interest rate 4% and $C = 2.2$ trillion (i.e. thousands of billion) euros. Compute its duration and its market value V at $t = 0$, assuming a flat term structure at the interest rate $i = 0.5\%$. The Treasury decides to lower the duration by 1 year, substituting BTPs for a market value of 400 billion euros with zero-coupon bonds with same value. Compute the maturity T of the ZCB that reaches such a goal. (Answers: $V = 2.95$ trillions, $D = 8.61$ years, $T = 3.11$ years.)

Exercise 8.7. Mr. Smith wants to invest 10 000 euros and has three assets at disposal:

- an immediate perpetuity with constant semi-annual instalments;
- a 3-month BOT;
- a BTP with duration 9.5 years.

The term structure of interest rates is flat, with annual interest rate $i = 2\%$. Knowing that Mr. Smith wants to invest half of his money in the BOT and that the overall portfolio duration needs to be 10 years, determine the amounts V_p and V_{BTP} invested in the perpetuity and BTP respectively.
After setting up such a portfolio, Mr. Smith inherits 5000 euros and decides to invest them entirely in the BTP described above. Determine the duration D of the new portfolio of Mr. Smith.

Exercise 8.8. In a market, two assets are traded:

- a ZCB with maturity 10 years;
- a coupon bond with annual coupons and coupon rate 3%, maturity 5 years.

The term structure of interest rates is flat, with $i = 3\%$. Compute the durations of the two assets, expressed in years. Mr. Smith wants to invest 100 000 euros in this market, setting the duration of his portfolio to 7 years. Determine the quotas he needs to invest in the two assets.

Exercise 8.9. Consider a BTP just issued, with a nominal annual interest rate 4% and maturity 5 years. Compute the duration of the BTP if the term structure of interest rates is flat at $i = 3\%$. Consider a portfolio that invests 20% of its value in the BTP above, 50% in a 3-month BOT and the remaining in perpetuities with annual instalments. Compute the duration D' of such a portfolio.

Exercise 8.10. Consider the assets:

- **x**: ZCB with notional 11500 euros and maturity 6 months;
- **y**: coupon bond with semi-annual coupons, nominal annual interest rate 6%, notional 10000 euros and maturity 1.5 years.

Assuming a flat term structure of interest rates with $i = 5\%$, compute the prices and durations of the two assets. Consider an investment of 40 000 euros in a portfolio composed of α units of **x** and β of **y**. Compute α and *beta* such that the portfolio has duration 1 year.

9 Bonds indexed to interest rates

So far, we have considered contracts in which the amounts exchanged were fixed and certain. Often, in practice, cash flows are indexed to the evolution of financial quantities. In particular, we will consider contracts in which changes in the term structure of interest rates affect cash flows, i.e. where cash flows are linked to interest rates at later dates than the one at which the contract is signed. Notable examples of such contracts include indexed government bonds, like Italian CCTs (Certificati di Credito del Tesoro).

9.1 Unitary indexed zero coupon bond and indexed coupon

We define as ZCB_i the basic contract in this context, which is a contract signed at t promising the cash flow $X_{T,s} = 1 + j(T, s)$ at maturity s, where $j(T, s)$ is a certain periodic interest rate observed at time T, with $t < T \leq s$. Hence, $X_{T,s}$ is random until time T, in which the periodic interest rate $j(T, s)$ becomes known. The following theorem helps us understanding what is the time t-price of this random cash flow.

Theorem 9.1. Suppose no arbitrage holds. Then, the time t-price of the random cash flow $X_{T,s}$ is

$$P(t, X_{T,s}) = p(t, T).$$

Proof. Suppose on the contrary that $P(t, X_{T,s}) > p(t, T)$. Then, the following strategy:

a) buy at t a unitary ZCB maturing at T;

b) buy at t the ZCB_i;

c) reinvest at time T 1 until s at the period rate $j(T, s)$

is an arbitrage. Indeed, it returns the following cash flows:

	t	T	s
a)	-p(t,T)	1	0
b)	$P(t, X_{T,s})$	0	$-X_{T,s}$
c)	0	-1	$1 + j(T, s)$
Total	$P(t, X_{T,s}) - p(t, T) > 0$	0	0

Hence, we reached a contradiction. We reach a similar conclusion when considering $P(t, X_{T,s}) < p(t, T)$, setting up a symmetric arbitrage. Then, the theorem is proved. □

Holding the indexed ZCB is thus equivalent to having the ZCB maturing at time T and reinvesting it until s. This replicating strategy is called rollover strategy. Intuitively, the theorem implies that, at time t it must be equivalent to buy the ZCB with maturity T and receiving its payoff 1 at T and then reinvesting at the periodic rate $j(T, s)$ and to buy the ZCB_i, that will deliver $1 + j(T, s)$. In other words, both the ZCB with maturity T and the ZCB_i paying the cash flow $X_{T,s}$ have time-T value 1. Notice that, in this case, we have considered a case in which the indexation is based on a single value, $j(T, s)$, with s coinciding with the maturity of the bond. This is a particular case, which simplifies our calculations.

Remark 9.1. Even though the duration cannot be readily computed when assets have unknown cash flows, we exploit the fact that the value of the ZCB_i is equal to 1 at T with certainty to say that, for all instants $t < T$, its duration must be equal to that of a ZCB with maturity T, i.e. $T - t$. After T, then, uncertainty regarding $j(T, s)$ is resolved and the only cash flow to be paid will be the known quantity $1 + j(T, s)$ at time s. Hence, when $t \geq T$, duration will be equal to $s - t$.

We define the indexed coupon $c_{T,s}$ the value of the coupon representing the indexed part of the remuneration of the buyer of a ZCB_i, i.e.

$$c_{T,s} = X_{T,s} - 1 = j(T, s).$$

Even though it is unknown at time t, the indexed coupon will be positive, since $j(T, s)$ is. The previous theorem, together with the theorem of linearity of prices, allows us to compute its value:

$$P(t, c_{T,s}) = P(t, X_{T,s}) - p(t, s) = p(t, T) - p(t, s) > 0.$$

The value of the indexed coupon is then equivalent to the value of a contract that delivers two unitary cash flows, one positive at time T and a negative one at time s. The above formulas extend naturally to the case in which a spread, i.e. an additional fixed amount, usually expressed as a percentage of the notional, is added to the indexed coupon. If we call σ such spread, the value of a ZCB_i promising an indexed coupon $c_{T,s} = j(T, s) + \sigma$ is still equal to $p(t, T)$, while the value of the indexed coupon is

$$P(t, c_{T,s}) = P(t, X_{T,s}) + \sigma p(t, s) - p(t, s) = P(t, X_{T,s}) - (1 - \sigma)p(t, s).$$

9.2 Floating Rate Notes

In this section, we use the building blocks we defined in the previous section to determine the value of indexed contracts, such as Floating Rate Notes (or floaters). These are assets paying indexed coupons at regular intervals, on top of reimbursing the notional amount at maturity. As in the previous section, we will consider the case in which the indexation is performed based on a single rate observation and refers to a periodic interest rate whose period is equal to the distance between two consecutive coupon payments. Assume no spread is paid on top of $j(T, s)$. We denote the notional, as usual, with C, and with I_k, $k = 1, ..., n$ the indexed coupons.

Let us first evaluate this contract at a date $t \leq t_0$, where t_0 indicates the time at which the first indexed coupon is determined, hence before issuance. The time-t value of the floating rate note paying cash flows $\mathbf{x} = \{I_1, I_2, ..., I_n + C\}$ at dates $\mathbf{t} = \{t_i\}_{\{i=1,..,n\}}$ is thus equal to:

$$P(t, \mathbf{x}) = \sum_{k=1}^{n} P(t, I_k) + Cp(t, t_n).$$

Each indexed coupon $I_k, k = 1, ..., n$ is equal to

$$I_k = Cj(t_{k-1}, t_k) = C\left[\frac{1}{p(t_{k-1}, t_k)} - 1\right],$$

whose time-t value is

$$P(t, I_k) = C\left[p(t, t_{k-1}) - p(t, t_k)\right].$$

It is thus easy to see that

$$P(t, \mathbf{x}) = \sum_{k=1}^{n} C\left[p(t, t_{k-1}) - p(t, t_k)\right] + Cp(t, t_n) = Cp(t, t_0).$$

Hence, the value of the coupon bond before issuance is equal to that of a ZCB with notional C and maturity equal to the date of issuance.
Suppose now $t \geq t_0$. Notice first that, ad date t, the value of the first coupon, I_1 is known and equal to $Cj(t_0, t_1)$. The value of the bond is thus equal to

$$\begin{aligned} P(t, \mathbf{x}) &= I_1 p(t, t_1) + \sum_{k=2}^{n} P(t, I_k) + Cp(t, t_n) = \\ &= I_1 p(t, t_1) + C\sum_{k=2}^{n}(p(t, t_{k-1}) - p(t, t_k)) + Cp(t, t_n) = \\ &= I_1 p(t, t_1) + Cp(t, t_1) = (I_1 + C)p(t, t_1). \end{aligned}$$

Hence, the floating rate note is equivalent to a zero coupon bond with notional $I_1 + C$ and maturity t_1.

Remark 9.2. Notice that the value of the floating rate note is equal to C at issuance and, more generally, is equal to C any time a coupon is paid. Hence, the value of the floating rate note is known with certainty at any coupon payment date.

Remark 9.3. Since the value of the floating rate note is known at any coupon payment date, we can replicate the reasoning we applied to the indexed ZCB to determine the duration of the floating rate note. Indeed, duration will be equal to $t_k - t_{k-1}$ at any t_k and, more generally, equal to $t - t_{k-1}$ at any t.

If a spread is applied to the coupon payments, and each coupon is equal to $j(t_{k-1}, t_k) + \sigma$, the floating rate note is a bond that pays:

- the notional C at maturity;
- the indexed coupons at the dates $t_1, ..., t_n$;
- fixed coupons at the dates $t_1, ..., t_n$.

Hence, its price can be seen as the sum of the prices of the standard floating rate note, with cash flows $\mathbf{y} = \{I_1, ..., C + I_n\}$ and the value of an annuity $\mathbf{a}$ paying n instalments of amount σC. Hence:

$$P(t, \mathbf{x}) = P(t, \mathbf{y}) + P(t, \mathbf{a}).$$

Analogously, the duration of such bond will be equal to:

$$D(t, \mathbf{x}) = \frac{P(t, \mathbf{y})}{P(t, \mathbf{x})} D(t, \mathbf{y}) + \frac{P(t, \mathbf{a})}{P(t, \mathbf{x})} D(t, \mathbf{a}).$$

Notice that both the price and the duration will be greater than those of a floating rate note not paying a spread over the indexation.

Example 9.1. The Italian government regularly issues floating rate notes, called CCTeu. Standard maturity at issuance is 7 years. The indexed coupons are paid every 6 months and are linked to the 6-month Euribor rate (plus a spread which varies with the issuance). The last issuance, for instance, placed via an auction at the end of April, promises the payment of an annual spread of 1.10% over the 6-month Euribor rate.

9.3 Interest Rate Swaps

Plain vanilla Interest Rate Swaps are financial instruments that allow one party to exchange a regular stream of fixed cash flows for a floating (variable) one.

Indeed, they allow to pay/receive a fixed interest rate over a notional amount while receiving/paying a floating interest rate over the same notional. Interest Rate Swaps are traded "over-the-counter", i.e. they are customized transactions in which the transaction is performed via brokers that receive commissions. They are now a widespread and liquid instrument, used in the inter-bank market in particular.
Let the notional of the contract be C. Such notional is never exchanged. It is indeed a reference quantity on which actual contract cash flows are computed. One party, at predetermined dates, $k = 1, ...n$ pays (receives) a fixed coupon iC, where i is the coupon rate, and receives (pays) the indexed coupon I_k. The series of fixed coupons is the fixed leg, while the series of indexed coupon is the floating leg. Denoting with $\mathbf{s}$ the vector of net cash flows of the swap, at inception, the fixed coupon rate i is chosen so that the contract is fair: $P(t_0, \mathbf{s}) = 0$. Indeed, the swap can be considered as the difference between a coupon bond CB (with coupon rate i) and a floating rate note FRN with coupons I_k. Indeed, for the fixed rate receiver:

$$P(t_0, \mathbf{s}) = 0 \implies P(t_0, CB) - P(t_0, FRN) = 0$$

$$\sum_{k=1}^{n} iCp(t_0, t_k) + Cp(t, t_n) - C = 0.$$

Hence, it follows that:

$$i = i_{SW}(0, n) = \frac{1 - p(t_0, t_n)}{\sum_{k=1}^{n} p(t_0, t_k)}. \tag{9.1}$$

Hence, depending on the maturity of the contract (n) the IRS rate will be different. Given the liquidity of the IRS market, taking the rates of the swaps of different maturities prevailing on the market, it is possible to derive a term structure of interest rates. It is then possible, exploiting equation (9.1), to establish a link between swap rates and spot rates. Indeed, assuming to know the IRS rates for n maturities, it is possible to recover the spot rates with the corresponding n maturities, since:

$$i_{SW}(0, k) \sum_{l=1}^{k} p(0, l) = 1 - p(0, k)$$

$$i_{SW}(0,k)\sum_{l=1}^{k-1} p(0,l) + (1+i_{SW})p(0,k) = 1$$

$$p(0,k) = \frac{1 - i_{SW}(0,k)\sum_{l=1}^{k-1}}{1 + i_{SW}(0,k)}.$$

Starting from $p(0,1)$ it is possible to determine, iteratively applying the above relations, the term structure of spot rates from the swap rates. Indeed, it is necessary to know all the n swap rates between 0 and n to be able to recover the corresponding spot rates.

Remark 9.4. In standard IRS contracts, the fixed leg and the variable leg are paid at different intervals. Usually, the fixed leg is paid annually, while the variable one is paid every 3 or 6 months. Notice that the periodicity of the variable leg payment does not affect the formula we have derived for i. Indeed, the price of the floating rate note $P(t_0, FRN)$ is C independently of the distance between two consecutive payments. Indeed, the distance between two payments of the fixed leg influences i.

9.4 Adjustable-rate Mortgages (ARM)

An adjustable-rate mortgage is a mortgage in which the periodic interest rate applied to residual debt to obtain the interest is indexed and varies throughout the life of the loan. Indeed, the n instalments of this mortgage will be obtained as

$$R_k = C_k + D_{k-1}j(k-1,k),$$

where C_k are the amortization quotas, and the interest quotas depend on the indexed rate $j(k-1,k)$. In such mortgages usually the succession of amortization quotas is usually fixed and known, and the condition $S = \sum_{k=1}^{n} C_k$ must hold. Denoting with $\mathbf{r}$ the succession of cash flows R_k, k=1,...n, the value of the ARM at t will be

$$\begin{aligned} V(t,\mathbf{r}) &= \sum_{k=1}^{n} V(t,R_k) = \sum_{k=1}^{n} V(t,C_k + D_{k-1}c_{t_{k-1},t_k}) = \\ &= \sum_{k=1}^{n} \left[C_k p(t,t_k) + D_{k-1}\left[p(t,t_{k-1}) - p(t,t_k)\right]\right], \end{aligned}$$

which can be shown to be equal to

$$V(t,\mathbf{r}) = (S + I_1)p(t,t_1),$$

if $t < t_1$ and

$$V(t,\mathbf{r}) = (D_k + I_{k+1})p(t,t_{k+1}),$$

for all $t_k < t < t_{k+1}$. At payment dates, moreover, the value will be equal to residual debt.

Exercise 9.1. On April, 8th, 2017, an investor buys a CCT without spread, with maurity August, 8th, 2025 at a price $P = 102$ for 100 euros notional. Knowing that the term structure of interest rates is described by

$$i(0,t) = 4\% + 1\%t,$$

compute the duration of the CCT and the value of the first coupon I_1. Compute the swap rate of a plain vanilla contract with semi-annual payments and maturity 2 years.

Exercise 9.2. Consider a floating rate note, issued at $t = 0$, paying at $t_2 = 3$ the amount $100 * I_{2,3}$, where $I_{2,3}$ is the indexed coupon known at t_2. Compute the no-arbitrage price P of the bond at $t = 0$, if the ZCBs with maturity 2 and 3 years trade at 97 and 95.5, respectively. In the same market, a ZCB with maturity 1 year is quoted at 98.5. Compute the swap rate of a plain vanilla contract with maturity $T = 3$ years, having annual payments. (Answers: $P = 1.5; i_{SW} = 1.546\%$).

Exercise 9.3. A CCT with no spread has maturity 1 year and 5 months, in a market where the term structure of interest rates is flat at $i = 5\%$. Knowing that the price (over 100 euros notional) is $P = 103.2$, compute the next coupon and the duration of the CCT expressed in years. Answer the same question if the CCT pays a spread of 80 bp per coupon. (Answers: $I_1 = 5.31945, D = 0.4167; I_1' = 3.7668, D' = 0.4276$).

Exercise 9.4. The Treasury has issued a CCT with maturity 1.5 years, semi-annual coupons and a spread of 4.5% on each coupon. If the term structure of interest rates is flat at the interest rate $i = 5.5\%$, compute the price P and the duration of the CCT an instant after issuance. A bank invests 150 mln euros in such CCT, but decides to sell the position after three days. Determine, approximatively, the revenue following the transactions, if the term structure of interest rate has grown by 18 bp for every maturity at the end of the three days. (Answers: $P = 112.8, D = 0.5557, \Delta V \approx -150000$ euros)

10 Additional exercises

Exercise 10.1. An entrepreneur seeks for a loan S= 400 000 euros, to be reimbursed in 4 annual instalments at the annual interest rate $i = 5\%$. The first instalment is 1/4 the initial debt, the amortization quota of the second instalment is 1/4 the initial debt and the residual debt value after the payment of the third instalment is 30% the initial debt. Fill the amortization plan.

Exercise 10.2. Mr. Smith wants to invest S=380000 euros to obtain an interest $I = 60000$ euros. He has two options:

1. Invest in an asset that gives him an annual compound interest rate $i_C = 5\%$;

2. Invest in an asset that returns a simple interest $i_S = i_C + 1\%$.

Determine the investment horizon in the two cases, T_C and T_S, expressed in years. Which of the two options will he choose?

Exercise 10.3. Mr. Brown wants to take on a loan of S euros, to be reimbursed via a non-standard amortization plan with 3 instalments paid every 4 months. The annual interest rate is 4% and the instalments are decreasing: each rate is 80% the preceding one. Knowing that Mr. Brown can afford to pay a maximum instalment $R_{max} = 20000$, compute the maximum initial debt S that Mr. Brown can demand, S_{max}. Fill the amortization plan when $S = S_{max}$.

Exercise 10.4. Consider the investment of $S = 25000$ euros in an asset returning a 6-month interest rate 6.2% according to an exponential law. Compute the time T, in years, necessary to triple the investment and the IRR of the transaction, expressed on an annual basis. Answer the same question in case the asset returns the same interest rate, according to a linear law. (Answers: With the exponential law, $T = 18.2634$ semesters; IRR= 12.7844%; with the linear law, T=32.2581 semesters ; IRR=7.0488%).

Exercise 10.5. Consider the financial transaction $\mathbf{x}/\mathbf{t} = \{-100, 60, 42\}/\{0, 6, 12\}$ and time is expressed in months. Compute the IRR i^* of the transaction on an annual basis. Compute the value Δx that needs to be added to the second cash flow, keeping fixed vector $\mathbf{t}$, to obtain an IRR that is 0.5% greater than i^* (i.e. equal to $i^* + 0.5\%$). (Answers: $i^* = 2.86\%, \Delta x = 0.35$).

Exercise 10.6. Consider an individual who wants to take on a loan S=150 000 euros, to be reimbursed in 3 6-month instalments at an interest rate $i = 8\%$. The amortization is non-standard and consists in:

- having the residual debt after the second payment equal to 40000 euros;
- having a first instalment equal to 50 000 euros.

Fill the corresponding amortization plan.

Exercise 10.7. Miss Smith holds two securities:

- $\mathbf{a}$ is an immediate annuity due, with 11 constant instalments R=100 euros paid every quarter;
- $\mathbf{a}'$ is a deferred perpetuity, with deferral period of length 5 years and annual instalments R'=10 euros.

Compute the present values of the two securities, according to an exponential law characterized by an instantaneous interest rate $\delta = 0.021$ years^{-1}. Miss Smith wants to sell the two securities at a price $P = W(0, \mathbf{a}) + W(0, \mathbf{a}')$. Determine the internal rate of return of the transaction composed of the sale at price P of the portfolio, expressed on an annual basis. Compute the value of such a transaction after 6 months and decompose it in its accumulated and residual values, M and V. (Answers: W(0,a)= 1071.65; W(0,a')=424.24; $i^* = 2.1222\%$; W=0, M=1210.10 euro; V=-1210.10).