Economic models and the relevance of “chaotic regions”: An application to Goodwin’s growth cycle model

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In this paper, we argue that Pohjola’s one-dimensional, discrete-time version of Goodwin’s growth cycle model is based on assumptions that conflict with the “symbiotic-conflictual’’ spirit of the model. It is shown that when the assumption about the dynamical real wage is modified, in contrast with Pohjola’s opinion, the likelihood of chaotic solutions does not increase. In particular, when a discrete-time Phillips curve is considered, the model becomes two-dimensional, but admits chaotic solutions only for parameter values which are not within economically reasonable values.

Keywords: chaos, simulation, growth cycles

1. Introduction

An important source of advance in economic dynamic modelling has often been the interaction with other fields of scientific research. This is clearly testified to by numerous and well-known examples, such as the use of the van der Pol oscillator to explain the persistence of economic fluctuations (Goodwin [11]), the use of Lotka–Volterra equations to formalize the growth cycles resulting from the symbiotic contradictions of capitalism (Goodwin [12]) and, more recently, the use of chaos theory to represent the irregularity of economic fluctuations.

In particular, in the latter case, the time lag between the first applications in other disciplines (e.g. May [18,19]) and the first applications in economics (e.g. Benhabib and Day [4,5], Stutzer [23]) has been unusually short. The resulting economic contributions are very heterogeneous in that some of them are based on equilibrium assumptions and maximization principles, whereas others adopt a disequilibrium and behavioural view of the economy. However, they can be seen as a whole because they all use the new concepts of dynamical systems theory to show that the complexity and irregularities of the dynamics of actual capitalist economies can be explained by the nonlinearities that are endogenous in their functioning, rather than by means of linear equations with unexplained exogenous shocks (as in the mainstream approach to business cycle modelling).

Although it may be too early to establish the impact that all this will have on the way economists think about dynamical problems, we can certainly say that the immediate effect of it was to increase the number of low dimension economic dynamical models formulated in discrete time. This, however, appears to be in contrast with what has happened and is happening in other scientific disciplines, where it is safe to say that the most important use of maps is that of assisting in the study of continuous time, higher dimensional models (see Alliwood et al. [1, p. 273]).

A typical example is that of the two-dimensional (2D) map known as the Hénon map (see, for example, Hilborn [14, pp. 240–241]):

$$\boldsymbol{f}(x, y) = (a - x^2 + by, x), \tag{1}$$

which was introduced by Hénon as a simplified model of the Poincaré map for the Lorenz (3D) differential equations model.

In other words, the procedure is the following: first, the model is formulated in terms of *all* required variables (dimensions); second, in order to make progress in the study of its dynamics, attention is restricted to a lower-dimensional version of the model (for example to its Poincaré map) that still contains the "essential" information (Hilborn [14, p. 181]).

In the economic applications, on the other hand, the model is often first manipulated in such a way that its final equation is a 1D difference equation. The commonly used justification for such a revision of a given model is to view it only as a kind of "*first step*" in the right direction, to be followed by the relaxation of some of the simplifying assumptions. The reason for this – it is often added – is that, once the possibility of chaotic solutions has been shown for a simplified (1D) version of a given model, then we can expect the model to produce chaotic solutions for a larger range of parameter values (i) if we introduce in it a further nonlinearity, and/or (ii) if we increase its dimensionality.

In this paper, we develop a simple exercise with the purpose of disproving such a justification.

First, in section 2, we introduce a specific example, namely the article published in 1981 by Pohjola [21], where the author analyses a 1D, discrete-time version of Goodwin's growth cycle (LVG) model [12], originally framed in terms of Lotka–Volterra 2D differential equations. In doing this, our purpose is to show that, in contrast with Pohjola's view, the consideration of a nonlinear bargaining equation (rather than a linear one as in the simplified case considered by the author) does not increase the likelihood of chaotic solutions at all. Second, in section 3 – after having argued that Pohjola's analysis of Goodwin's model as a 1D map is based on assumptions that conflict with the "symbiotic-conflictual" spirit of the original model – it is shown, again in contrast with Pohjola's view, that once his simplifying assumptions are relaxed, the resulting 2D, discrete-time dynamical system admits chaotic solutions only for parameter values which are not within economically reasonable ranges.

Some concluding and summarising remarks are then given in section 4.

2. The LVG model as a 1D map

2.1. *Pohjola's original elaboration*

A low-dimensional nonlinear dynamical system capable of generating chaotic behaviour – such as an equation of the logistic type[1)] or such as (1) of the previous section – can be easily obtained by suitably revising a given dynamical model of the economy (Sordi [22]).

In his (1981) article, Pohjola [21] follows this "modelling strategy", taking as his starting point Goodwin's 1967 celebrated growth cycle model [12] (originally framed in the Lotka–Volterra 2D system of differential equations).

In the Pohjola elaboration of the model, *apart from the change in the time concept* (from continuous to discrete), all but one assumption are the same as in the original version of the model (Goodwin [12, p. 54], Pohjola [21, p. 28]).

In both versions of the model, it is assumed that (see also Sordi [22, p. 12]): (i) technical progress is labour-augmenting at a constant rate,(ii) the labour supply grows at a constant rate, (iii) there are only two factors of production (labour and capital), both homogeneous and non-specific, (iv) all quantities are real and net, (v) all wages are consumed, all profits saved and invested, and (vi) the capital–output ratio is constant. Writing all the relations in discrete-time terms, we thus have

$$\frac{q_{t+1}}{l_{t+1}} = (1+\alpha)\frac{q_t}{l_t} \qquad t, \alpha \geq 0, \tag{2}$$

$$n_{t+1} = (1+\beta)n_t, \qquad t, \beta \geq 0, \tag{3}$$

$$k_{t+1} - k_t = (1-u_t)q_t, \tag{4}$$

$$\sigma = \frac{k_t}{q_t}, \qquad t, \sigma > 0, \tag{5}$$

where q is output, l employment, n labour supply, k capacity (capital), w real wage, and where $u = wl/q$ denotes the workers' share of total product.

Writing $\upsilon (= l/n)$ for the employment rate, from (2)–(5) one obtains

$$\frac{\upsilon_{t+1}}{\upsilon_t} = 1 + \frac{1 - \sigma g - u_t}{\sigma(1+g)}, \tag{6}$$

where $g = \alpha + \beta + \alpha\beta \geq 0$.

The only modification introduced by Pohjola consists in replacing Goodwin's real wage equation (which is a Phillips curve in real terms) by an equation which makes the *level* (as opposed to the *relative change*) of the *workers' share of total product* (as opposed to the *real wage*) depend positively on the employment rate.

1) By this, we mean an equation such as $x_{t+1} = f(x_t, a) = ax_t(1 - x_t)$, with a [0, 4] and x_t [0, 1], for all t, or, more generally, $x_{t+1} = F(x_t, a)$, where F is a unimodal map function, with the unique maximum that increases in the control parameter a. See Brock and Dechert [7, pp. 2211–2212].

Thus, the two versions of the model can be characterized and distinguished in the following way:

(I) Goodwin's original version (in discrete time): equation (6) together with

$$\frac{w_{t+1} - w_t}{w_t} = f(\upsilon_t), \quad -1 < f(0) < 0, \; f'(\upsilon) > 0, \; f''(\upsilon) > 0, \quad t, \tag{7}$$

where Goodwin chooses to work with a linear approximation of the function $f(\upsilon)$ such as (see figure 1(a))

$$f(\upsilon) \approx -\gamma + \rho\upsilon, \quad 0 < \gamma < 1, \; \rho > 0. \tag{8}$$

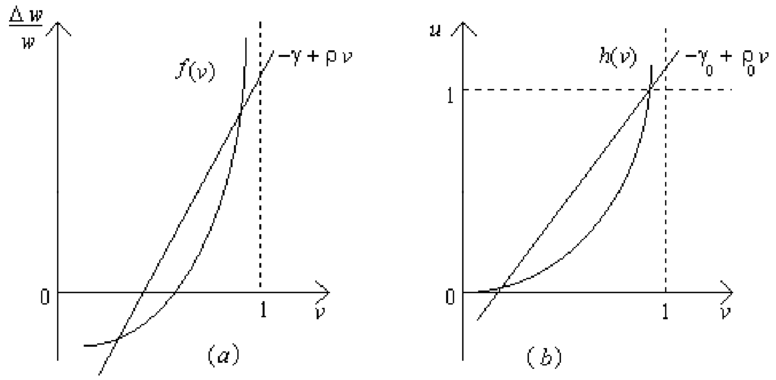

Figure 1. The bargaining equation (a) in Goodwin's model and (b) in Pohjola's version of the model.

(II) Pohjola's version: equation (6) together with

$$w_t = h(\upsilon_t)\frac{q_t}{l_t}, \; h(0) = 0, \; h'(\upsilon) > 0, \; h''(\upsilon) > 0, \quad \upsilon, \tag{9}$$

where Pohjola chooses to work with a linear approximation of the function $h(\upsilon)$ such as (see figure 1(b))

$$h(\upsilon) \approx -\gamma_0 + \rho_0\upsilon, \; \gamma_0 > 0, \; \rho_0 > 0. \tag{10}$$

He also suggests that the nature of the results would not change by using a nonlinear approximation such as

$$h(\upsilon) \approx \gamma_1\upsilon^{\rho_1}, \; \gamma_1 > 0, \; \rho_1 > 1. \tag{11}$$

As is easily seen, linear approximation plays a crucial role in both cases: in (I) it allows Goodwin to obtain the Lotka–Volterra prey–predator case, in (II) it allows Pohjola to obtain a logistic equation.

Concentrating for the time being on the second case, we notice that due to the modification of the model introduced by Pohjola, inserting (9)–(10) into (6), one easily obtains[2)]

$$x_{t+1} = (1 + r)x_t(1 - x_t), \tag{12}$$

where $x_t = r\upsilon_t/[(1 + r)\bar{\upsilon}_2]$, $r = (1 - \sigma g + \gamma_0)/[\sigma(1 + g)]$ and where $\bar{\upsilon}_2 = (1 - \sigma g + \gamma_0)/\rho_0$ is the only non-trivial fixed point of the difference equation (expressed in terms of the original variable). As is easy to see, (12) is a logistic equation to which all available mathematical results are readily applicable (Pohjola [21, pp. 30–35], Sordi [22, pp. 14–15]).

The spirit of Pohjola's exercise is self-evident and stated explicitly by the author when he says [21, p. 30] that, choosing the linear approximation (10), his purpose is ". . . to gain analytical simplicity at the expense of empirical realism since (the) aim is not to construct an empirically realistic model but to show that even the simplest nonlinear models can have complicated solutions." Moreover, he clearly considers his analysis only as a "*first step*", as is testified to by two passages of his article: first, when he says [21, p. 30] that the choice of the nonlinear approximation (11) rather than the linear approximation (10) would avoid some unrealistic features of the model, but would make the analysis more difficult without, however, changing the nature of the results[3)]; second, when he says [21, p. 36] that we should expect the model to produce chaotic solutions for a larger range of parameter values if we replaced the bargaining equation (9) with a Phillips curve. As stressed by Pohjola himself, the reason for this is that, in the latter case, we should need *two* variables, rather than only *one*, to describe growth paths.

It is now our intention to show that, whereas for mathematical reasons this is surely true in general, it is not so for the specific model we are considering, *once one remembers that it is not an abstract mathematical model, but rather is taken to represent a well-defined economic situation*. In doing this, we first analyse the case in which the nonlinear bargaining equation (11) is introduced into the model, whereas we postpone to the following section the case in which the Phillips curve (7) is considered. In both cases, the crucial element of our analysis will be the consideration of all restrictions that for economic reasons must be imposed on the values of the parameters and variables of the model.

2.2. A "more nonlinear" elaboration

We first notice that, in (11), the restrictions we have given for the values of the two parameters are stricter than those considered by Pohjola [21, p. 30], who simply requires that the two parameters be positive. The reason for this is that, to satisfy the

[2)] See Pohjola [21, pp. 29–30] for calculations.

[3)] In his concluding remarks [21, p. 36], Pohjola also adds that the linear approximation (10) is in no way necessary for the qualitative results obtained in his paper.

spirit of Goodwin’s model – according to which labour never becomes the limiting factor in production – the function $h(\upsilon)$ must be a positive, increasing, convex function of υ for $0 < \upsilon < 1$, growing fast enough when full employment is approached so as to have [4)]

$$h'(\upsilon) = \rho_1\gamma_1\upsilon^{\rho_1-1} > 0, \qquad \upsilon \in (0,1) \tag{13a}$$

$$h''(\upsilon) = (\rho_1 - 1)\rho_1\gamma_1\upsilon^{\rho_1-2} > 0, \qquad \upsilon \in (0,1) \tag{13b}$$

$$h(1) = \gamma_1 > 1 + \sigma. \tag{13c}$$

As we see, (13a) and (13b) require that $\gamma_1 > 0$ and $\rho_1 > 1$.

Then, inserting (9) into (6) and choosing (11) rather than (10), we obtain the following nonlinear 1D difference equation:

$$\upsilon_{t+1} = A\upsilon_t(1 - B\upsilon_t^{\rho_1}) = G(\upsilon_t), \tag{14}$$

where $A = (1+\sigma)/[\sigma(1+g)] > 0$ and $B = \gamma_1/(1+\sigma) > 1$.

Given the economic definition of the variables, we must ensure that $\upsilon_t < 1$ for all t. To this end, we notice that $G(\upsilon)$ attains a maximum at $\upsilon = \upsilon_{max}$ such that

$$\upsilon_{max} = \left[\frac{1}{(1+\rho_1)B}\right]^{1/\rho_1} = \left[\frac{1+\sigma}{(1+\rho_1)\gamma_1}\right]^{1/\rho_1} < 1, \quad \text{always true.}$$

For this value of the variable, we must also have

$$G(\upsilon_{max}) = A\left[\frac{1}{(1+\rho_1)B}\right]^{1/\rho_1}\frac{\rho_1}{1+\rho_1} < 1,$$

a condition that is satisfied if

$$A < \frac{(1+\rho_1)^{1+1/\rho_1}B^{1/\rho_1}}{\rho_1}. \tag{15}$$

Moreover, to ensure that $\upsilon_t > 0$ for all t, given that $G(\upsilon)$ intersects the υ-axis at $\upsilon_1 = 0$ and at $\upsilon_2 = (1/B)^{1/\rho_1}$, we must have

$$G(\upsilon_{max}) < \left(\frac{1}{B}\right)^{1/\rho_1},$$

from which [5)]

$$A < \frac{(1+\rho_1)}{\rho_1}(1+\rho_1)^{1/\rho_1} = A_+. \tag{16}$$

As is easy to check, condition (16) ensures that condition (15) is also satisfied.

[4)] In particular, condition (13c) implies that at full employment, workers’ real income claim would be higher than the obtainable maximum (which is current output plus the existing capital stock).

[5)] In our simulations, we have chosen $\rho_1 = 2$, so that $A_+ \approx 2.598$.

The fixed points of (14) are

$$\bar{\upsilon}_1 = 0,$$

$$\bar{\upsilon}_2 = \left(\frac{A-1}{AB}\right)^{1/\rho_1} = \left(\frac{1-\sigma g}{\gamma_1}\right)^{1/\rho_1},$$

so that the following condition must also be satisfied:

$$B > \frac{A-1}{A}, \quad \text{always true,} \tag{17a}$$

$$A > 1. \tag{17b}$$

Given condition (17b), we have

$$G'(0) > 1.$$

Moreover,

$$G'(\bar{\upsilon}_2) = 1 - \rho_1(A-1).$$

Thus, the trivial fixed point is always unstable, whereas the non-trivial fixed point is stable if

$$|1 - \rho_1(A-1)| < 1,$$

i.e., if

$$1 < A < 1 + \frac{2}{\rho_1}. \tag{18}$$

2.3. *Bifurcation analysis*

Considering only values of A greater than one, i.e., taking account of condition (17b), it is now not difficult to show that, as was the case for Pohjola's original version of the model, there exists a period-doubling route to chaos, as the parameter A is increased. The problem, however, is that, taking account of all restrictions we have found on the values of the parameters, the solution to this "more nonlinear" version of the model is chaotic only for a rather small range of parameter values.[6)]

To show this, let us notice that, given condition (18), with $\rho_1 = 2$ and $B = 1.2$, the non-trivial fixed point is stable for $1 < A < 2$. For values of A in this range, the convergence to the non-trivial equilibrium point is monotonic when $1 < A < 1.5$ and cyclical when $1.5 < A < 2$. This is shown in figure 2(a) and figure 2(b), respectively. Then, as A is increased, we have the period-doubling sequence summarised in table 1. Three examples – a 4-period cycle, an 8-period cycle and a chaotic solution – are shown in figures 3(a), 3(b) and 4, respectively.

From table 1, it follows that the *economically relevant* "chaotic region" is restricted to the interval $2.3 < A < A_+ = 2.598$. The reason for this is that although the model shows chaotic solutions also for values of A greater than the critical values A_+ (for example for $A = 2.6$ as shown in figure 5), the latter are *economically irrelevant*

[6)] Or, in other words, for the majority of admissible parameter values, the solution is periodic!

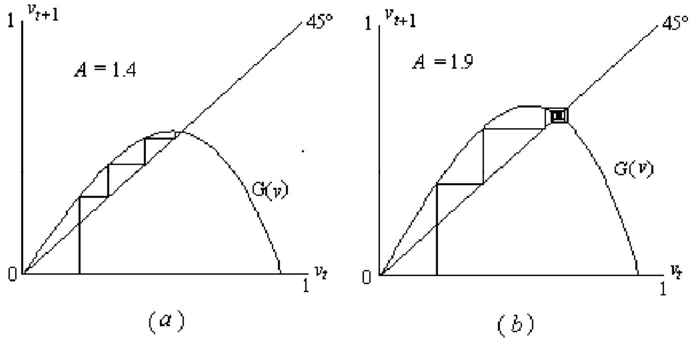

Figure 2. (a) A monotonic and (b) a cyclical convergence to the non-trivial equilibrium point.

Table 1

Period-doubling sequence for parameter *A*.

Dynamic behavior	Values of *A*
2-period cycle	2.000–2.236
4-period cycle	2.237–2.287
8-period cycle	2.288–2.298
16-, 32-, 64…-period cycle	2.299–2.300
Chaotic behaviour	2.301–2.598

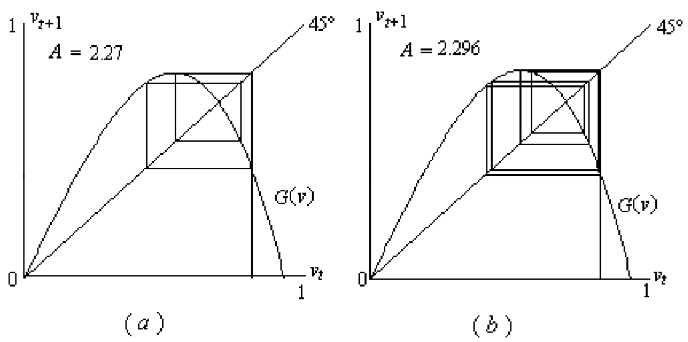

Figure 3. (a) A 4-period cycle and (b) an 8-period cycle.

because – for about half of the simulation period – they imply negative values for both the employment rate and the workers' share of total product.

A look at the bifurcation diagrams of the two models (see figure 6 for the version of the model with the nonlinear approximation (11) and figure 7 for the version with

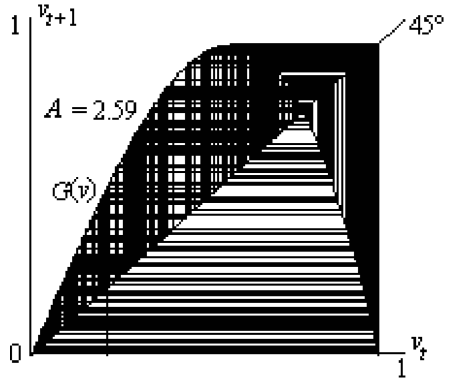

Figure 4. An example of an "economically relevant" chaotic solution to Pohjola's model with the nonlinear bargaining equation (11).

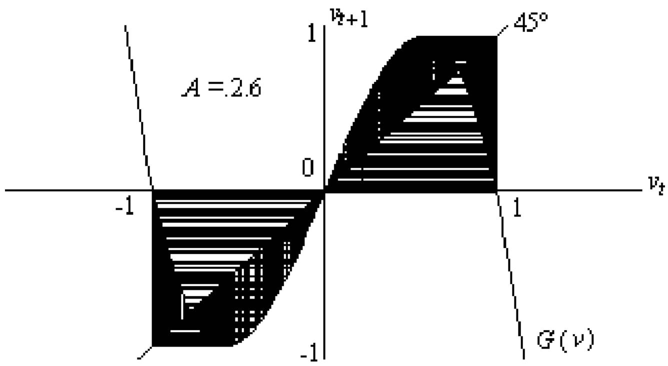

Figure 5. An example of an "economically irrelevant" chaotic solution to Pohjola's model with the nonlinear bargaining equation (11).

the linear approximation (10)) shows clearly that the "chaotic region" is larger in the first case. However, only half of it or so is economically relevant.

Thus, although the likelihood of chaotic solutions in the two versions of the model is not directly comparable,[7] there are certainly no reasons to conclude that it is higher in the " *more nonlinear*" version of the model!

This ends the first part of our exercise. As we will see, however, a more important aspect is to comment in detail on the assumptions which allow Pohjola to reduce Goodwin's original 2D dynamical system to a 1D equation of the logistic type.[8]

[7] This is due to the fact that, as we have seen, Pohjola's results are expressed in terms of the parameter $r = (1 - \sigma g + \gamma_0)/[\sigma(1 + g)]$, ours in terms of the parameter $A = (1 + \sigma)/[\sigma(1 + g)]$.

[8] On this problem, see also Sordi [22].

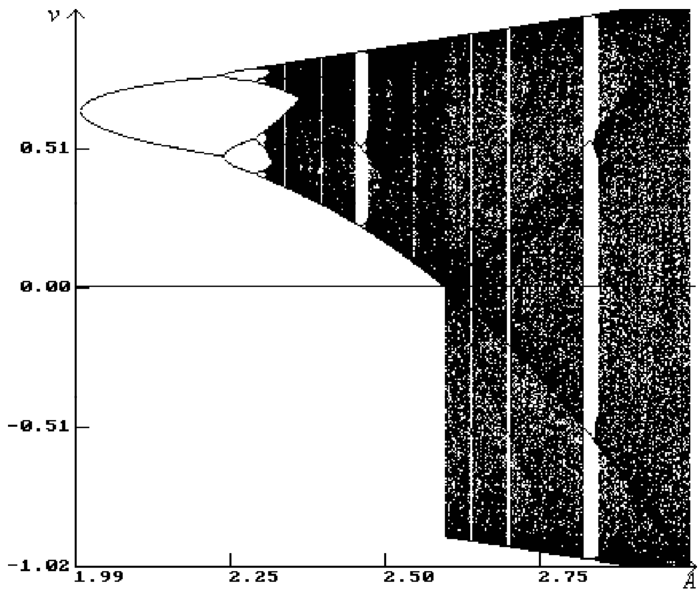

Figure 6. The bifurcation diagram of Pohjola's model with the nonlinear bargaining equation (11).

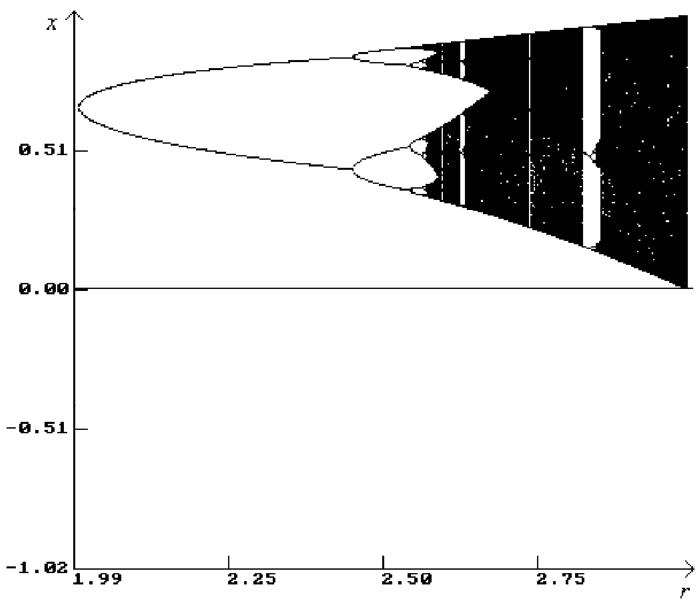

Figure 7. The bifurcation diagram of Pohjola's model with the linear bargaining equation (10).

3. The LVG model as a 2D map

As we have already stressed in the introduction, one of the two justifications given by Pohjola [21, p. 36] for the consideration of a revised, 1D discrete-time version of Goodwin's model is that, once the model has been shown to possess chaotic solutions for the simplest case, we should expect it to possess chaotic solutions for a larger range of parameter values in the case in which we replace (9) with the original Phillips curve (7). Our purpose is now to show that this is not true once one remembers the *economic meaning* of the variables.

First of all, we first notice that – as is well known and testified to by its analogy with Lotka–Volterra's theory of the cyclical growth of two competing species of fish (partly complementary, partly hostile) – the *most economically relevant* characteristic of Goodwin's model is its ability to represent in a complete and very simple model the symbiotic contradictions of capitalism, leading to perpetual *class-conflict* cycles through the expansion and contraction of the size of the reserve army of labour.[9] This result – given the simplifications introduced by Goodwin in order to meet the Lotka–Volterra case[10] – appears to have been obtained with the *simplest possible structure*. It is hard, then, to imagine a further simplification (such as the reduction of the dimension of the dynamical system) which is able to preserve the original (symbiotic–conflictual) spirit of the model.

Keeping this in mind, it is not difficult to understand that with the formulation chosen by Pohjola, the main economic characteristic of Goodwin's model appears to be lost. To this end, it is useful to carry on the analogy with biological situations[11] and to remember, first, that the logistic equation is used in biology to represent *single species* situations; second, that the mathematical description in terms of difference equations is in biology taken to be appropriate only in the case in which the single population under consideration grows at discrete intervals of time and in which *generations are completely non-overlapping*. As a consequence, it is hard to imagine how any kind of interaction between workers and capitalists could be represented in terms of such a framework.

The problem, however, is that the consideration of the original Phillips curve (7) instead of equation (9) certainly increases the dimensionality of the resulting dynamical system but not the likelihood of chaos. This can be shown by using both analytical arguments and numerical simulations.

[9] See Goodwin [12, pp. 57–58], where we find a (more or less Marxian) description of the dynamics of the economic system in terms of the variations and interaction of the capitalists' share of total product and the employment rate. For a further investigation of the problem, see also the contributions in Goodwin et al. [13, Part I, pp. 1–72].

[10] The most important of such simplifications, as stressed in Fitoussi and Velupillai [9] and Velupillai [25, pp. 80–86], is the assumption of equilibrium in the goods market.

[11] On this aspect, see Beddington et al. [3], Holton and May [15], Marotto [17, p. 200], May [18,19].

3.1. Qualitative analysis of the model

We first notice that, inserting (7) into (6), one obtains the following dynamical system:

$$\begin{aligned} \upsilon_{t+1} &= [A_1 - B_1 u_t]\upsilon_t = F(\upsilon_t, u_t), \\ u_{t+1} &= A_2[1 + f(\upsilon_t)]u_t = G(\upsilon_t, u_t), \end{aligned} \tag{19}$$

where $A_1 = (\sigma + 1)/[\sigma(1 + g)] > 0$, $B_1 = 1/[\sigma(1 + g)] > 0$ and $A_2 = 1/(1 + \alpha) > 0$.

The fixed points of (19) are $(\bar{\upsilon}_1, \bar{u}_1) = (0, 0)$ and $(\bar{\upsilon}_2, \bar{u}_2)$, where the coordinates of the non-trivial fixed point are such that

$$\begin{aligned} f(\bar{\upsilon}_2) &= \frac{1 - A_2}{A_2} = \alpha, \\ \bar{u}_2 &= \frac{A_1 - 1}{B_1} = 1 - \sigma g. \end{aligned} \tag{20}$$

Thus, given the economic meaning of the variables, we must have[12)]

$$0 < f^{-1}(\alpha) < 1, \tag{21a}$$

$$0 < 1 - \sigma g < 1. \tag{21b}$$

An evaluation of the Jacobian matrix of the system, at the fixed points in question, allows one to study the local stability of the two fixed points, even without specifying a functional form for $f(\upsilon)$.

We have

$$\boldsymbol{J} = \begin{bmatrix} A_1 - B_1 \bar{u} & -B_1 \bar{\upsilon} \\ A_2 f'(\bar{\upsilon})\bar{u} & A_2[1 + f(\bar{\upsilon})] \end{bmatrix}. \tag{22}$$

Thus, at $(\bar{\upsilon}_1, \bar{u}_1)$,

$$\mathrm{Tr}\ \boldsymbol{J}_1 = A_1 + A_2[1 + f(0)] > 0,$$

$$\mathrm{Det}\ \boldsymbol{J}_1 = A_1 A_2[1 + f(0)] > 0,$$

whereas at $(\bar{\upsilon}_2, \bar{u}_2)$,

$$\mathrm{Tr}\ \boldsymbol{J}_2 = 2 > 0,$$

$$\mathrm{Det}\ \boldsymbol{J}_2 = 1 + B_1 A_2 f'(\bar{\upsilon}_2)\bar{\upsilon}_2 \bar{u}_2 > 0.$$

Moreover, writing $P(\lambda)$ for the characteristic polynomial,

$$P(\lambda) = \lambda^2 - (\mathrm{Tr}\ \boldsymbol{J})\lambda + (\mathrm{Det}\ \boldsymbol{J}),$$

the three necessary and sufficient conditions for (local) stability of system (19) can be expressed as (see Azariadis [2, pp. 58–67], Gandolfo [10, pp. 58–59])

[12)] As is easy to check, (21b) requires that $A_1 > 1$.

$$P(1) = 1 - \mathrm{Tr}\,\boldsymbol{J} + \mathrm{Det}\,\boldsymbol{J} > 0, \tag{23a}$$

$$P(-1) = 1 + \mathrm{Tr}\,\boldsymbol{J} + \mathrm{Det}\,\boldsymbol{J} > 0, \tag{23b}$$

$$1 - \mathrm{Det}\,\boldsymbol{J} > 0. \tag{23c}$$

For the case of the trivial equilibrium point, it can immediately be seen that[13)]

$$P(1) = (1 - A_1)[1 - A_2 - A_2 f(0)] = -\frac{1 - \sigma g}{\sigma(1+g)}\left[\frac{\alpha - f(0)}{1+\alpha}\right] < 0,$$

so that condition (23a) is always violated.

We also have

$$\Delta_1 = (\mathrm{Tr}\,\boldsymbol{J}_1)^2 - 4\mathrm{Det}\,\boldsymbol{J}_1 = \{A_1 - A_2[1 + f(0)]\}^2 > 0, \quad \text{always},$$

so that the two characteristic roots are *real*. Moreover, they are *both positive* and such that

$$\lambda_1 = A_2[1 + f(0)] = \frac{1 + f(0)}{1+\alpha} < 1, \quad \lambda_2 = A_1 = \frac{\sigma + 1}{\sigma(1+g)} > 1.$$

Thus, as in Goodwin's original formulation of the model (see Medio [20, pp. 36–37]), the origin of the coordinate axes is a *saddle point*.

Considering next the case of the non-trivial fixed point, it is easy to check that

$$1 - \mathrm{Det}\,\boldsymbol{J}_2 = -B_1 A_2 f'(\bar{\upsilon}_2)\bar{\upsilon}_2 \bar{u}_2 < 0,$$

so that condition (23c) is always violated.

We also have

$$\Delta_2 = (\mathrm{Tr}\,\boldsymbol{J}_2)^2 - 4\mathrm{Det}\,\boldsymbol{J}_2 < 0, \tag{24}$$

so that we can conclude that the non-trivial fixed point is an *unstable focus*.

3.2. Numerical simulation

Specifying a functional form for $f(\upsilon)$, we can then employ – together with these analytical arguments concerning the nature of the fixed points – some numerical simulations to show that the likelihood of chaotic solutions in this model is really very low!

For the sake of simplicity, let us choose the linear approximation (8). In this case, the dynamical system of the model becomes

$$\upsilon_{t+1} = A_1\upsilon_t - B_1\upsilon_t u_t,$$
$$u_{t+1} = A_3 u_t + B_3\upsilon_t u_t,$$

where $A_3 = A_2(1 - \gamma) > 0$ and $B_3 = \rho A_2 > 0$.

13) This follows from (21b) and the fact that the function $f(\upsilon)$, for $\upsilon = 0$, is negative.

For system (25), the non-trivial equilibrium is equal to

$$(\bar{\upsilon}_2, \bar{u}_2) = \left(\frac{1 - A_3}{B_3}, \frac{A_1 - 1}{B_1}\right),$$

so that the conditions in (21) now read

$$\begin{aligned} 1 - B_3 < A_3 < 1, \\ 1 < A_1 < 1 + B_1. \end{aligned} \tag{26}$$

Using only *economically meaningful parameter values*, i.e., values that satisfy conditions (26),[14)] the simulations show very clearly the nature of the two fixed points we established on the basis of the qualitative analysis of the dynamical system. As is shown in figures 8 and 9, starting from a (υ, u)-combination near the unstable focus,

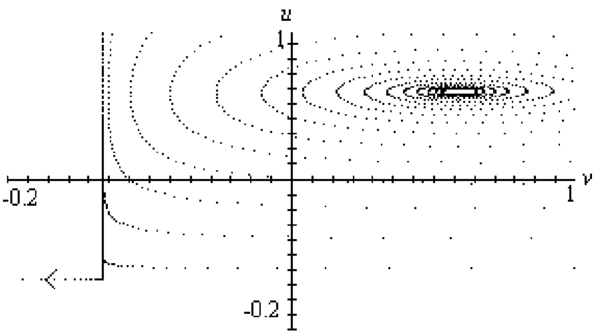

Figure 8. Phase diagram of the discrete-time, 2D version of the model.

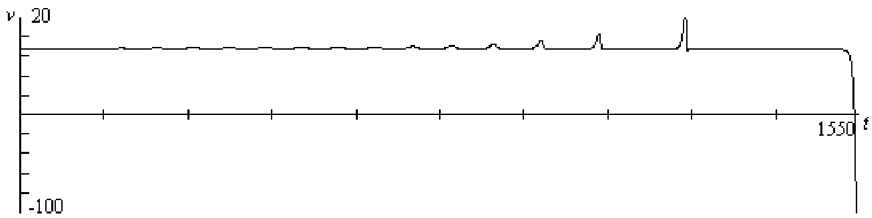

Figure 9. Trajectory of the discrete-time, 2D version of the model.

the system shows (anticlockwise) fluctuations of increasing amplitude, which eventually enter the basin of attraction of the saddle point. When this happens, either (by chance!) the trajectory follows one of the two asymptotic curves forming the *inset* of the saddle (see Thompson and Stewart [24, pp. 204–206] or (more likely!) it moves

[14)] One possibility, and one which we have followed, is to use for the various parameters the values given in Blatt [6, pp. 208].

away from it, for example along the $u = 0$-axis (the *onset* of the saddle). Clearly, *both* cases are completely uninteresting from the point of view of economic theory.[15)]

It is not difficult, however, to understand why Pohjola's expectations about the likelihood of chaotic solutions are not fulfilled.

The problem is that, as is well known and as is also testified to by some recent economic applications of chaos theory (see, for example, Delli Gatti et al. [8]), the reason for the greater likelihood of chaotic solutions in higher dimensional dynamical systems is the fact that, for 2D, the dynamics may become chaotic also via an infinite sequence of Hopf bifurcations, a phenomenon which has no counterpart in 1D maps (Holton and May [15], Lauwerier [16]). Without going into details, for which we refer to the above quoted contributions, it is, however, easy to understand why this cannot happen in the present case.

Roughly, the occurrence of a Hopf bifurcation requires:

(i) the existence of a pair of (conjugate) complex eigenvalues;

(ii) the existence of values of the parameters for which the moduli of the complex pair of eigenvalues are equal to one.

At the non-trivial equilibrium point, given (26), (i) certainly holds. However, from the characteristic equation, we find that

$$\lambda_{1,2} = 1 \pm i \sqrt{(A_1 - 1)(1 - A_3)},$$

where $i = \sqrt{-1}$. Thus, the modulus of both eigenvalues is *always greater than one* so that condition (ii) never holds.

The only case in which we have been able to find seemingly chaotic solutions – which are shown in figures 10 and 11 – is for A_1 and A_3 both equal to one. This is a degenerate case in which, among other things, the variable also assumes negative values and both equilibria are trivial.

4. Conclusions

In this paper, we have developed an exercise that, notwithstanding its simplicity, allows us to draw a twofold conclusion.

On the one hand, it seems to strongly suggest the necessity of discussing in more detail than is usually the case – and eventually relaxing! – those assumptions which merely serve the purpose of keeping the order of the dynamical system "low" and which may be unsatisfactory from the point of view of economic theory. On the other hand, it shows that the "discretization" of existing continuous-time, highly aggregate

[15)] We can also add that, using a wide range of values of the parameters, we have not been able to find a chaotic attractor.

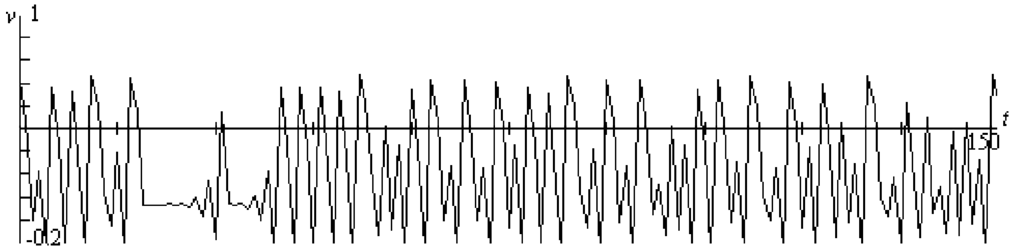

Figure 10. One irregular trajectory in the discrete-time, 2D version of Goodwin's 1967 model.

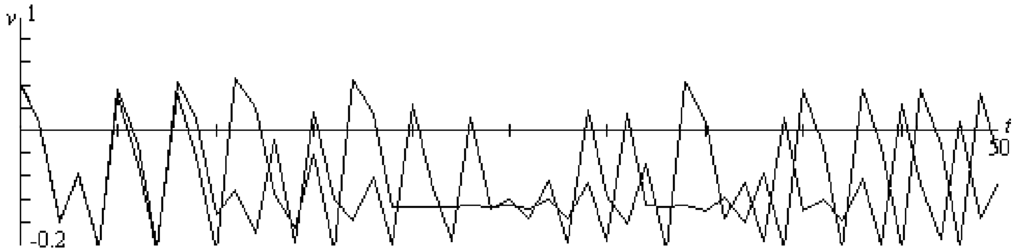

Figure 11. Two trajectories showing sensitive dependence on initial conditions.

models is not only theoretically unsatisfactory, but can also be useless from a more practical point of view. Indeed, when one considers a 2D version of the model that more closely resembles the original framework of Goodwin's work, the result is that the dynamics appear to be chaotic only for parameter values that are not within economically reasonable bounds.

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