CEJOR (2001) 9: 97–117

CEJOR

Growth cycles when workers save

A reformulation of Goodwin's model along Kaldorian-Pasinettian lines

Serena Sordi

Dispartimento di Economia Politica, Università di Siena, Piazza San Francesco 7, 53100 Siena, Italy (e-mail: sordi@unisi.it)

Abstract The purpose of this paper is to study the influence of workers' savings on the dynamics of Goodwin's growth cycles. This is done by noting, along Kaldorian-Pasinettian lines, that if workers save, they then hold part of the capital and earn some profits, which vary over the cycle. Thus, a correct specification of such a case requires the consideration of an extra variable – the share of capital held by workers. It is shown that, without having to impose any special condition on the values of the parameters, a Hopf-Bifurcation analysis establishes the possibility of persistent and bounded cyclical paths for the resulting 3-dimensional dynamical system. The paper concludes with an investigation of the possibility of further bifurcations as a route to more complex behaviour.

1 Introduction

Over the years, outside mainstream macroeconomics, Goodwin's growth cycle model [25] has emerged as a powerful and fruitful "*system for doing macrodynamics*". Yet, despite the more than one hundred contributions that have tried to generalize it in all possible directions, there is an aspect that still seems to deserve further investigation, namely, the problem of

---

Send offprint requests to: Serena Sordi

* *The author is indebted to Fabio Petri who some time ago had the basic idea and sketched the mathematics required for the local stability analysis of the dynamical system of a model similar to the one investigated in this paper. Comments and suggestions by Massimo Di Matteo and two anonymous referees are also gratefully acknowledged. Needless to say, responsibility for the views expressed herein and for any remaining errors is mine alone.*

the proper way of relaxing Goodwin's extreme assumptions about savings behaviour and wage rate dynamics.

As is well known, Goodwin [25, p. 54] crucially assumed (i) a *classical savings function*, according to which workers consume all their wages and capitalists save and invest all their profits, and (ii) a *Phillips curve in real terms*[1] ($\hat{w} = f(v) = -\gamma + \rho v$, $df/dv = f'(v) = \rho > 0$, $\gamma > 0$) according to which the growth rate of real wages (w) varies *linearly* with the employment rate (v). A possible interpretation of (i) is to say that Goodwin assumed that capitalists' and workers' propensities to save (s_c and s_w, respectively) are different and such that $\Delta s = s_c - s_w > 0$. Then, as a limiting and simpler case, he chose to work with $s_c = 1$ and $s_w = 0$.[2] An immediate generalization, thus, is to consider the case in which $0 \leq s_w < s_c \leq 1$ and this is indeed what a number of authors have tried to do over the years.[3] With regard to assumption (ii), on the other hand, it is safe to say that Goodwin meant it only as a first approximation, required — we could add — to obtain in his model a dynamical system of the Lotka-Volterra type. It is therefore easy to think of possible generalizations of this assumption too. Different authors have considered a number of different more general cases, for example, with a nonlinear f (Velupillai [48]) or with the function f depending not only on v, but also on its rate of change[4] (Cugno-Montrucchio [6, pp. 97-98], Sportelli [44, pp. 43-44])

$$\hat{w} = f(v, \hat{v}) \tag{1}$$

such that $\partial f(v, \hat{v}) / \partial v = f_v(v, \hat{v}) > 0$ and $\partial f(v, \hat{v}) / \partial \hat{v} = f_{\hat{v}}(v, \hat{v}) > 0$.

Both kinds of generalization have important implications for the dynamics of the model.

To show this, let us start from the original version of the model (OVM, hereafter). In it, given the other basic assumptions — according to which the labour force (n) and the productivity of labour ($q/l = a$) both grow

[1] For any variable x, $\dot{x}$ indicates its time derivative (dx/dt), while $\hat{x}$ indicates its growth rate ($\dot{x}/x$).

[2] This interpretation seems correct given that Goodwin himself writes [25, p. 54] that his assumption could be relaxed to constant proportional savings, "this changing the numbers but not the logic of the system".

[3] The first author to mention the possibility of incorporating into the model a positive propensity to save out of wages was Atkinson [2] in 1969. Other contributions that have tried to generalize the model in this direction are Ferri-Greenberg [13], Fitoussi-Velupillai [14], Flaschel-Krüger [15], Glombowski-Krüger [21], [22], [23], [24], van der Ploeg [38], [39], [40], [41], Sportelli [44], Sportelli-Cagnetta [45] and Velupillai [49], to mention only a few.

[4] This seems to be the case considered by Phillips himself (see [37, p. 299]). Above all, however, starting with Izzo [27] and then Desai [7], the Phillips curve in generalizations of Goodwin's model has been modified by writing it in *money* rather than *real* terms and then introducing into the model an equation for price dynamics. Although we will not consider this kind of generalization of the original model, we will have something to say about it at the end of the paper.

exponentially in time at a rate equal to β and α respectively and the capital-output ratio ($k/q = \sigma$) is constant — (i) and (ii) are all that is needed in order to obtain the dynamical system in $v = l/n$ (the *employment rate*) and $u = wl/q$ (the *share of wages*)

$$\hat{v} = g - g_n = \left(\frac{1}{\sigma} - g_n\right) - \frac{1}{\sigma}u \tag{2}$$

$$\hat{u} = f(v) - \alpha = -(\alpha + \gamma) + \rho v \tag{3}$$

where $g = \hat{q} = \hat{k} = I/k = S/k = (1-u)/\sigma$ and $g_n = \alpha + \beta$ (the *natural* rate of growth).

The unique positive equilibrium of system (2)-(3) is (v^e, u^e), where $v^e = (\gamma + \alpha)/\rho$ and $u^e = 1 - \sigma g_n$. Given that the roots of the characteristic equation at this equilibrium point are pure imaginary, the local stability analysis cannot be used to decide on the type of dynamics of the original system. In the case in which the parameters of the model are such that $(1/\sigma) - g_n > 0$, however, (2)-(3) is equivalent to the dynamical system of the Lotka-Volterra prey-predator model (see Goodwin [25, pp. 54-55], Gandolfo [20, pp. 449-464] and Medio [31, pp. 34-38]). Thus, we can conclude that the equilibrium point is a *centre*; in other words, that the model describes persistent fluctuations of the variables v and u around the equilibrium, the amplitude of which fully depends on initial conditions.

This qualitative feature of the model does not change if, *leaving unaltered the rest of it*, we generalize (i) along the lines described above. In this case, the rate of growth of output is equal to $g = [(s_c/\sigma) - (\Delta s/\sigma)u]$ so that (2) becomes

$$\hat{v} = \left(\frac{s_c}{\sigma} - g_n\right) - \frac{\Delta s}{\sigma}u \tag{4}$$

Clearly, if we choose the same linear f as in the OVM and if $(s_c/\sigma) - g_n > 0$, equations (3) and (4) still form a dynamical system of the Lotka-Volterra type. The only difference in comparison with the OVM is that now the coordinates of the centre are (v^e, u^{ee}), where $u^{ee} = (s_c - \sigma g_n)/\Delta s$.

Having noticed this, however, things change drastically if, in the attempt to go further in our generalization of the model, we introduce in (3) the more general formulation of the Phillips curve (1) so as to obtain

$$\hat{u} = f(v, \hat{v}) - \alpha = F(v, u) - \alpha \tag{5}$$

where $F(v, u)$ is such that $F_v(v, u) = f_v(v, \hat{v}) + f_{\hat{v}}(v, \hat{v})(\partial\hat{v}/\partial v) = f_v(v, \hat{v}) > 0$ and $F_u(v, u) = f_{\hat{v}}(v, \hat{v})(\partial\hat{v}/\partial u) = -f_{\hat{v}}(v, \hat{v})\Delta s/\sigma < 0$.

The non-trivial equilibrium of system (4)-(5) is now (v^{ee}, u^{ee}), where v^{ee} is that value of the employment rate for which $f(v^{ee}, 0) = \alpha$. Easy calculations show that, in this case, the characteristic equation at (v^{ee}, u^{ee}) is

$$\lambda^2 + \left\{\frac{\Delta s}{\sigma}u^{ee} f_{\hat{v}}(v^{ee}, 0)\right\}\lambda + \frac{\Delta s}{\sigma}v^{ee}u^{ee} f_v(v^{ee}, 0) = 0 \tag{6}$$

where $(\Delta s/\sigma)u^{ee} f_{\hat{v}}(v^{ee}, 0) > 0$ and $(\Delta s/\sigma)v^{ee}u^{ee} f_v(v^{ee}, 0) > 0$.

Thus, the two roots of (6) have negative real parts and we can conclude that, *whatever the sign of the discriminant*, the movement is convergent.[5]

Summing up, the first of the two extensions of the OVM we have considered simply "chang(es) the numbers, but not the logic of the system", the second destroys its cyclical features.

The purpose of this paper is to show that the conclusion is drastically different if, in jointly introducing into the OVM both a positive propensity to save by workers and the more general formulation of the Phillips curve (1), we take account of Pasinetti's criticism (e.g. [35], [36]) of Kaldor's theory of growth and income distribution (e.g. [28]). From such a "Kaldorian-Pasinettian" point of view, it follows that both extensions of Goodwin's model we have considered contain a "logical slip", consisting in an incomplete consideration of the implications that a positive propensity to save by workers has for the model. In particular (see Pasinetti [35, p. 106]) — as was the case for Kaldor's model — they overlook the fact that, if workers save a part of their income, they "must be allowed to own it" and to earn some profits from it (if not there would be no reason for them to save at all). This simple consideration has extremely important consequences in the case of Goodwin's model because it happens that the share of the capital stock owned by workers (or, alternately, the share owned by capitalists) and the profits they earn from it are not constant, but rather vary with the two other variables of the model (v and u) over the cycle. Thus, the consideration of a positive propensity to save by workers not only implies a change in the equation for the rate of growth of output — as in (4) — but also an increase in the dimensionality of the dynamical system of the model.[6] Due to this, a persistent cyclical movement (limit cycle) can emerge. Moreover, the latter may prove to be the first step on a route to more complex (irregular) behaviour as one or more of the parameters of the model are varied.

---

[5] Indeed, this is the case considered by Lorenz [30, pp. 67-73] to show the structural instability of Goodwin's model; to prove, in other words, that the behaviour of its dynamical system is very sensitive to variations in the functional structure such as a change from $f(v)$ to $F(v,u)$.

[6] To the best of my knowledge, among the contributions that I have cited in footnote 3 above, only van der Ploeg (e.g., [40]) has introduced the hypothesis of differential savings into Goodwin's model in a manner that tries to take account of Pasinetti's criticism. In a sense the analysis we will develop, although based on a simpler extension of Goodwin's model, puts together two of van der Ploeg's insights, one taken from [40] — where, though the analysis is confined to local stability, the dimensionality of the dynamical system increases, as in our case, because of variation in the workers' share of the capital stock — and another one, taken from [41], where, since the workers' share of the capital stock remains constant, the increase in the dimensionality of the model, from 2 to 3, is due to a different factor (namely, the fact that a more dynamic view of technical change is adopted, with the capital-output ratio varying with the cost of labour over the cycle). However, as in this paper, the Hopf bifurcation theorem is applied to the resulting dynamical system.

To show this, the rest of the paper is organized as follows. In Section 2 the alternative formulation of Goodwin's model we have just described is introduced and the 3-dimensional (3D) dynamical system of the modified model is derived and analysed with respect to local stability. Then, in Section 3, we attempt to show — by use of the Hopf Bifurcation Theorem (HBT, hereafter) with s_w as the bifurcation parameter — that the model can produce persistent cyclical behaviour. The section ends with an investigation — by means of numerical simulation — of the possibility of further bifurcations on a route to chaos. Finally, some concluding and summarizing results are given in Section 4.

2 The model when workers save: a Kaldorian-Pasinettian formulation

2.1 Derivation of the dynamical system

Re-adapting to the case the notation introduced in the previous section, we can take account of Pasinetti's point about the implications of a positive propensity to save of workers by writing

$$\begin{aligned} q &= wl + P_w + P_c \\ S_w &= s_w\,(wl + P_w) = s_w\,(wl + rk_w) \\ S_c &= s_c P_c = s_c r k_c \\ r &= \frac{P}{k} = \frac{1-u}{\sigma} \end{aligned}$$

where P_w (P_c), S_w (S_c), and s_w (s_c) are workers' (capitalists') profits, savings and propensity to save respectively, k_w (k_c), capital held by workers (capitalists), $P = P_w + P_c$, total profits, $k = k_w + k_c$, total capital stock and r, the rate of profit.

For this modified version of the model, the rate of growth of output is equal to $g = [(s_w/\sigma) + (\Delta s/\sigma)(1-u)\varepsilon]$ — where ε is the proportion of capital held by capitalists ($\varepsilon = k_c/k$) — so that (4) becomes

$$\hat{v} = \left(\frac{s_w}{\sigma} - g_n\right) + \frac{\Delta s}{\sigma}\,(1-u)\,\varepsilon \tag{7}$$

Thus, inserting in (5)

$$\hat{u} = f\,(v, \hat{v}) - \alpha = F_1(v, u, \varepsilon) - \alpha \tag{8}$$

Finally we find

$$\hat{\varepsilon} = \hat{k}_c - g = \frac{\Delta s}{\sigma} - \frac{s_c}{\sigma}u - \frac{\Delta s}{\sigma}\,(1-u)\,\varepsilon \tag{9}$$

2.2 Singular points

The dynamical system (7)-(9) has a unique interior singular point (equilibrium) $(v^*, u^*, \varepsilon^*)$ such that

$$\frac{s_w}{\sigma} - g_n + \frac{\Delta s}{\sigma}(1-u^*)\varepsilon^* = 0$$
$$f(v^*, 0) - \alpha = 0$$
$$\frac{\Delta s}{\sigma} - \frac{s_c}{\sigma}u^* - \frac{\Delta s}{\sigma}(1-u^*)\varepsilon^* = 0$$

Thus $v^* = v^{ee}$ as in the previous elaboration of the model, whereas we now have

$$u^* = 1 - \frac{\sigma g_n}{s_c} \tag{10}$$

$$\varepsilon^* = \frac{s_c(g_n - s_w/\sigma)}{\Delta s g_n} \tag{11}$$

It is easy to check that these equilibrium values of the modified version of Goodwin's model imply all the steady-state results that follow from Kaldor-Pasinetti's theory of growth and income distribution; in particular (see Pasinetti [35, p. 269], [36, pp. 121-122 and 127-128])

- in order to be economically meaningful (i.e., such that $0 < u^* < 1$ and $0 < \varepsilon^* < 1$), they require that

$$0 \leq s_w < \sigma g_n < s_c \leq 1 \tag{12}$$

  which is nothing other than the well-known condition that Kaldor's original model too must satisfy;
- they guarantee a steady-state growth of the system at a rate equal to the natural rate

$$g^* = \frac{s_w}{\sigma} + \frac{\Delta s}{\sigma}(1-u^*)\varepsilon^* = g_n$$

- unlike what was the case for the version of the model with differential savings we considered in the previous section, they imply the so-called *Cambridge equation*, according to which the steady-state rate of profit is determined by the natural rate of growth divided by the capitalists' propensity to save, *independently of anything else*

$$r^* = \frac{1-u^*}{\sigma} = \frac{g_n}{s_c}$$

It is worth stressing, however, that there is a basic difference between the two approaches. Indeed, whereas in the Kaldor-Pasinetti model the steady-state growth path continuously guarantees the full employment of labour, in the Goodwin model it is characterized by a positive (constant) rate of unemployment equal to $(1 - v^*)$. Moreover, and more importantly, it is possible to show that Goodwin's model, in the more general case of differential savings and *in the case in which condition (12) is satisfied*, admits closed orbits solutions (limit cycles). In other words, it describes persistent fluctuations of the variables rather than a convergence to the steady-state solution.

This can be rigorously established by applying to system (7)-(9) the HBT. As a preliminary step, we study the (local) stability of the model at the interior singular point.

2.3 Local stability analysis

Linearising system (7)-(9) at $(v^*, u^*, \varepsilon^*)$, we obtain

$$\begin{bmatrix} \dot{v} \\ \dot{u} \\ \dot{\varepsilon} \end{bmatrix} = \mathbf{J}^* \begin{bmatrix} v - v^* \\ u - u^* \\ \varepsilon - \varepsilon^* \end{bmatrix}$$

where

$$\mathbf{J}^* = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

Given the basic assumptions introduced in the previous section and condition (12), the Jacobian matrix $\mathbf{J}^*$ is such that *the signs of all its coefficients are unambiguously determined*

$$\begin{aligned}
a_{12} &= -(\Delta s/\sigma)\varepsilon^* v^* < 0 \\
a_{13} &= (\Delta s/\sigma)\,(1 - u^*)\, v^* > 0 \\
a_{21} &= F^*_{1v} u^* = f^*_{v} u^* > 0 \\
a_{22} &= F^*_{1u} u^* = -f^*_{\hat{v}}(\Delta s/\sigma)\varepsilon^* u^* < 0 \\
a_{23} &= F^*_{1\varepsilon} u^* = f^*_{\hat{v}}(\Delta s/\sigma)(1 - u^*)u^* > 0 \\
a_{32} &= -(s_c/\sigma)(s_w/\sigma g_n)\varepsilon^* < 0 \\
a_{33} &= -(\Delta s/\sigma)(1 - u^*)\varepsilon^* < 0
\end{aligned}$$

where the sign "*" indicates that all partial derivatives are evaluated at $(v^*, u^*, \varepsilon^*)$.

Thus, the characteristic equation of the dynamical system (7)-(9) becomes

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \qquad (13)$$

where

$$A = -\mathrm{tr}\,(\mathbf{J}^*) = -a_{22} - a_{33} > 0 \qquad (14)$$

$$\begin{aligned}
B &= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\
&= a_{22}a_{33} - a_{12}a_{21} - a_{23}a_{32} > 0 \qquad (15)
\end{aligned}$$

$$C = -\det\,(\mathbf{J}^*) = a_{21}a_{12}a_{33} - a_{21}a_{13}a_{32} > 0 \qquad (16)$$

so that the sign of $AB - C$ is undetermined

$$AB - C \gtreqless 0 \tag{17}$$

Given that — by (14), (15) and (16) — A, B, C are always positive, when condition (17) holds with the ">" sign, all Routh-Hurwitz (necessary and sufficient) conditions for the local stability of the equilibrium are satisfied.[7] This, as was the case for the second extension of the model we have considered in section 1 (see equations (4) and (5)), would be the "end of the story" and we could conclude that the model economy (locally) converges toward $(v^*, u^*, \varepsilon^*)$. However, given that condition (17) can also hold with the "$\leq$" sign, things are not so simple! Indeed, this implies that the dynamical behaviour of the model can drastically change, from the qualitative point of view, as one or more of the parameters vary.

Using s_w as the bifurcation parameter, our purpose is now to apply the HBT to show that persistent cyclical behaviour can emerge as s_w varies.

3 The dynamics of the model

3.1 Application of the Hopf Bifurcation Theorem

In the past few years, the HBT has often been utilized to prove — both in 2D, 3D and higher dimension continuous-time dynamical systems — the existence of closed orbits.

Restricting our attention to the applications of the HBT to 3D dynamical systems,[8] we notice that usually only the *existence part* of the theorem has been applied, the reason for this being that the *stability part* requires

[7] See, for example, Gandolfo [20, pp. 221-223].

[8] This is the case — we believe — in which the HBT really adds value to the standard theory (e.g., Poincaré-Bendixson theorem), that can by contrast be applied to show the existence of closed orbits only in the case of 2D systems. Although we will not consider them, the interested reader can find examples of applications of the HBT to 2D dynamical systems in Feichtinger [12], Flaschel-Groh [16], Kind [29], Medio [32], Semmler [43], Sportelli [44], and Torre [47]. Moreover, Benhabib-Nishimura [4], [5] and Franke-Asada [18] contain applications of the HBT to higher than 3D systems (nD, nD and 4D respectively). There are then a number of contributions - notably by Farkas and co-workers (see, for example, Farkas-Kotsis [11]) and more recently by Fanti and Manfredi (see, for example, Fanti-Manfredi [8], [9], [10]) - in which the HBT is applied to higher dimension extensions of the Goodwin model. We notice, however, that in the latter extensions (see, for example, Fanti-Manfredi [10, pp. 383 and 385]), the higher dimensionality is the result of the consideration of lagged relations - together with the application of the so-called "linear-chain trick" - rather than of a truly more general structure of the model. Finally, with regard to applications of the HBT to models formulated in discrete-time, we simply refer to Gandolfo [20, pp. 491-499] and Medio [33, pp. 67-69] and to the literature mentioned therein.

conditions (involving third order or even higher mixed partial derivatives) to which it is hard to give any economic interpretation.[9] What has hardly been noticed, by contrast, is that the simple application of the existence part of the theorem is not uninfluential for the specification of the model either. Indeed, as shown by the contributions I have just mentioned, it often requires assumptions that are introduced for no other reason than that of satisfying the conditions of the theorem.[10] In our application, this is not the case because, as shown in **Proposition 1** below, for the version of Goodwin's model we are considering, the application of the existence part of the HBT requires only that condition (12) — *resulting from Kaldor-Pasinetti's theory of economic growth and distribution* — is satisfied.

Proposition 1 *When condition (12) is assumed, there exists a critical value of the workers' propensity to save (s_{wH}) that satisfies it and is such that equation (13) evaluated at $(v^*, u^*, \varepsilon^*)$ has the following properties:*

(1.1) it possesses a pair of simple complex conjugate roots ($\lambda_{2,3} = \theta \pm i\omega$), that become pure imaginary at the critical value; in other words, at $s_w = s_{wH}$

$$A(s_{wH})B(s_{wH}) - C(s_{wH}) = 0$$

(1.2) the real parts of the complex roots cross the real axis at non-zero speed; in other words, at $s_w = s_{wH}$

$$\left.\frac{d\theta}{ds_w}\right|_{s_w = s_{wH}} \neq 0$$

Proof (1.1) Given (14)-(16), we can write

$$\begin{aligned} AB - C = \\ &\frac{\Delta s}{\sigma}\varepsilon^* f_{\hat{v}}^* \left[f_{\hat{v}}^* u^* + (1-u^*)\right]\left[\frac{\Delta s}{\sigma}u^*\frac{s_w}{\sigma}\varepsilon^* + \frac{\Delta s}{\sigma}\varepsilon^* u^* \frac{\Delta s}{\sigma}(1-u^*)\varepsilon^*\right] \\ &+\frac{\Delta s}{\sigma} f_v^* u^* \varepsilon^* \left(-v^*\frac{s_w}{\sigma} + f_{\hat{v}}^* u^* \frac{\Delta s}{\sigma}\varepsilon^* v^*\right) \end{aligned}$$

Thus $AB - C$ changes sign (see Appendix A)

$$AB - C \gtreqless 0$$

according as to whether

$$\begin{aligned} s_w \lesseqgtr \sigma &\left\{\frac{f_{\hat{v}}^* (s_c/\sigma)\left[f_{\hat{v}}^* u^* + (1-u^*)\right] + f_v^* f_{\hat{v}}^* u^* v^* / (1-u^*)}{f_{\hat{v}}^* (s_c/\sigma)\left[f_{\hat{v}}^* u^* + (1-u^*)\right] + f_v^* v^* + f_v^* f_{\hat{v}}^* u^* v^* / (1-u^*)}\right\} g_n \\ &= \sigma D g_n = s_{wH} \end{aligned} \tag{18}$$

---

[9] This is underlined, for example, in Asada [1], Benhabib-Miyao [3], Franke [17], Gandolfo [20, pp. 476-477], van der Ploeg [41] and Sasakura [42].

[10] See, for example, assumption (A.8) in Sasakura [42, p. 437] or assumptions 1 and 2 in Franke [17, pp. 246 and 250].

The denominator of the fraction in (18) is certainly always greater than the numerator, so that

$$D < 1 \rightarrow s_{wH} < \sigma g_n$$

Moreover

$$D > 0 \rightarrow s_{wH} > 0$$

This completes the proof of the first part of the Proposition.

(1.2) By using the so-called *sensitivity analysis* (see, for example, Gandolfo [20, pp. 475-48]), it is then easy to show that the second requirement of the Proposition is also met. First of all, we notice that the coefficients of the characteristic equation are such that

$$\begin{aligned}
A &= \left[\frac{f_{\tilde{v}}^{*} u^{*}}{\sigma\left(1-u^{*}\right)}+\frac{1}{\sigma}\right]\left(\sigma g_{n}-s_{w}\right)=a\left(\sigma g_{n}-s_{w}\right), a>0 \\
B &= \frac{u^{*}}{1-u^{*}}\left(\frac{f_{\tilde{v}}^{*} g_{n}+v^{*} f_{v}^{*}}{\sigma}\right)\left(\sigma g_{n}-s_{w}\right)=b\left(\sigma g_{n}-s_{w}\right), b>0 \\
C &= \left(\frac{f_{v}^{*} u^{*} v^{*}}{\sigma}\right)\left(\frac{s_{c}}{\sigma}\right)\left(\sigma g_{n}-s_{w}\right)=c\left(\sigma g_{n}-s_{w}\right), c>0
\end{aligned}$$

so that we also have

$$\begin{aligned}
\frac{\partial A}{\partial s_{w}} &= -a=\frac{A}{s_{w}-\sigma g_{n}}<0 \\
\frac{\partial B}{\partial s_{w}} &= -b=\frac{B}{s_{w}-\sigma g_{n}}<0 \\
\frac{\partial C}{\partial s_{w}} &= -c=\frac{C}{s_{w}-\sigma g_{n}}<0
\end{aligned}$$

Second, we know that, apart from $A > 0$, $B > 0$ and $C > 0$ that is always true, for $s_w = s_{wH}$, one also has $AB - C = 0$. Thus, when $s_w = s_{wH}$, one root of the characteristic equation (λ_1) is real negative, whereas the other two are a pair of pure imaginary roots ($\lambda_{2,3} = \theta \pm i\omega$, with $\theta = 0$). This means that we have

$$\begin{aligned}
A &= -\left(\lambda_{1}+\lambda_{2}+\lambda_{3}\right)=-\left(\lambda_{1}+2 \theta\right) \\
B &= \lambda_{1} \lambda_{2}+\lambda_{1} \lambda_{3}+\lambda_{2} \lambda_{3}=2 \lambda_{1} \theta+\theta^{2}+\omega^{2} \\
C &= -\lambda_{1} \lambda_{2} \lambda_{3}=-\lambda_{1}\left(\theta^{2}+\omega^{2}\right)
\end{aligned}$$

so that, differentiating with respect to s_w, in the case in which $\lambda_{2,3}$ are purely imaginary ($\theta = 0$), we obtain

$$\begin{aligned}
\frac{\partial A}{\partial s_{w}} &= -\frac{\partial \lambda_{1}}{\partial s_{w}}-2 \frac{\partial \theta}{\partial s_{w}} \\
\frac{\partial B}{\partial s_{w}} &= 2 \lambda_{1} \frac{\partial \theta}{\partial s_{w}}+2 \omega \frac{\partial \omega}{\partial s_{w}} \\
\frac{\partial C}{\partial s_{w}} &= -\omega^{2} \frac{\partial \lambda_{1}}{\partial s_{w}}-2 \lambda_{1} \omega \frac{\partial \omega}{\partial s_{w}}
\end{aligned}$$

or:

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & 2\lambda_1 & 2\omega \\ -\omega^2 & 0 & -2\lambda_1\omega \end{bmatrix} \begin{bmatrix} \partial\lambda_1/\partial s_w \\ \partial\theta/\partial s_w \\ \partial\omega/\partial s_w \end{bmatrix} = \mathbf{A} \begin{bmatrix} \partial\lambda_1/\partial s_w \\ \partial\theta/\partial s_w \\ \partial\omega/\partial s_w \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$$

such that at $s_w = s_{wH}$

$$|\mathbf{A}| = 4\omega(\lambda_1^2 + \omega^2) > 0 \qquad (19)$$

$$|\mathbf{A_{s_w}}| = 2a\omega^3 > 0 \qquad (20)$$

Given that, by Cramer's rule

$$\frac{|\mathbf{A_{s_w}}|}{|\mathbf{A}|} = \left.\frac{\partial\theta}{\partial s_w}\right|_{s_w = s_{wH}}$$

conditions (19) and (20) prove the second part of the Proposition.

Thus, system (7)-(9) admits closed orbits solutions (persistent and bounded fluctuations of the variables) for values of s_w in the neighbourhood of s_{wH}.

Before discussing at greater length the implications of this result, it is interesting to note that, *in the case in which the model has closed orbits solutions*, its cyclical behaviour is such that the u^*- and the ε^*-coordinate of the singular point are equal to the average values of the respective variables over a whole cycle.[11]

Proposition 2 *For the dynamical system (7)-(9) of Goodwin's model with differential savings, the average values of the income share of labour and of the share of capital owned by capitalists over a closed orbit are equal to the respective coordinates of the singular point (i.e., to u^* and ε^* respectively).*

Proof (see Gandolfo [20, pp. 463-464] for the analogous property of the original Lotka-Volterra model) To show this, let us rewrite equations (7) and (9) as

$$\frac{d}{dt}\log v = \frac{s_w}{\sigma} - g_n + \frac{\Delta s}{\sigma}\varepsilon - \frac{\Delta s}{\sigma}u\varepsilon$$

$$\frac{d}{dt}\log \varepsilon = \frac{\Delta s}{\sigma} - \frac{s_c}{\sigma}u - \frac{\Delta s}{\sigma}\varepsilon + \frac{\Delta s}{\sigma}u\varepsilon$$

from which, integrating over a period T equal to the period of the oscillations

$$\left(\frac{s_w}{\sigma} - g_n\right)T + \frac{\Delta s}{\sigma}\int_0^T \varepsilon\, dt - \frac{\Delta s}{\sigma}\int_0^T u\varepsilon\, dt = 0 \qquad (21)$$

$$\frac{\Delta s}{\sigma}T - \frac{s_c}{\sigma}\int_0^T u\, dt - \frac{\Delta s}{\sigma}\int_0^T \varepsilon\, dt + \frac{\Delta s}{\sigma}\int_0^T u\varepsilon\, dt = 0 \qquad (22)$$

[11] As is well known, this is a property that is also satisfied by the original model. With regard to v^*, it is not possible to draw an analogous conclusion because we have not specified a functional form for the generalized Phillips curve.

From (21)-(22)

$$\left(\frac{s_w}{\sigma} - g_n\right) T + \frac{\Delta s}{\sigma} \int_0^T \varepsilon \, dt = -\frac{\Delta s}{\sigma} T + \frac{s_c}{\sigma} \int_0^T u \, dt + \frac{\Delta s}{\sigma} \int_0^T \varepsilon \, dt$$

from which, on average over a cycle

$$\frac{1}{T} \int_0^T u \, dt = 1 - \frac{\sigma g_n}{s_c} = u^*$$

Then, inserting in (21)

$$\frac{1}{T} \int_0^T \varepsilon \, dt = \frac{s_c \left(g_n - s_w/\sigma\right)}{\Delta s g_n} = \varepsilon^*$$

This completes the proof of the Proposition.

Moving on to discuss the result contained in **Proposition 1**, we first notice that the nonlinearity of the generalized Phillips curve does not play any role in the proof of it we have given above. This means that we can expect the result to hold even when — as in the numerical simulations that follow — we take a linear Phillips curve; in other words, even when the model is reduced to its "intrinsic" nonlinear structure. Second, the bifurcational approach we have used appears to be very useful in that it has allowed us to go beyond the conventional steady state results one easily obtains from local stability analysis.[12] As we have seen, although for a low propensity to save by workers — less than the critical value s_{wH} and in the limit equal to zero — the positive steady-state solution of the model is (locally) stable, increasing such a propensity destabilizes the model. However, this does not lead to an unrealistic situation with fluctuations of ever increasing amplitude, neither does it require the use of the "saddle-point trick". Rather, through an Hopf bifurcation, (HB, hereafter) the result seems to be that of persistent and bounded fluctuations of the employment rate, the income share of wages and the proportion of capital held by workers and capitalists. Having proved only the existence part of the HBT, however, we do not know whether the closed orbits will arise for $s_w < s_{wH}$, where the equilibrium is locally stable and the closed orbits are repelling (*sub-critical HB*) or for $s_w > s_{wH}$, where the equilibrium is locally unstable and the closed orbits are attracting (*super-critical HB*).[13] The numerical example and computer simulations that follow, however, seem to give evidence that the super-critical is in fact the case that prevails.

---

[12] On this point, see also Benhabib-Miyao [3, p. 591].

[13] It is worth being aware that there also exists the possibility of *degenerate* HB's, which, however, are only the border case between sub- and super-critical HB. See Strogatz [46, pp. 252-253].

3.2 Bifurcations and route to chaos: some numerical simulations

Traditionally, limit cycle behaviour in extensions of Goodwin's growth cycle model has been the result of the quest for *structural stability.* Examples of this approach can be found in Medio [31, pp. 40-53], Funke [19] and van der Ploeg [41], to mention only a few. Rather that going into the details of the discussion of all the problems associated with this important property of dynamical systems, we prefer to take a different point of view and to stress that both the original Goodwin model and the extensions just mentioned (*including ours!*) are — in a sense — unsatisfactory. The reason for this is that they all imply that fluctuations of the variables of the model are *periodic.*[14] On the contrary, *it is a fact* that, although bounded, the economic fluctuations observed in the real world are highly irregular (aperiodic). These two observations, however, are reconciled when it happens that the Hopf bifurcation, the existence of which we have just proved, turns out to be *the first step on the route of the system from regularity to a chaotic regime.*[15]

To end this paper, we try now to give an idea of how this may happen by resorting to numerical simulations. In order to do this, we need a functional form for $f(v, \hat{v})$. Choosing to work with the simplest case in which f is additive and linear in both variables, we can write

$$f(v,\hat{v}) = -\gamma + \rho v + \delta \left[\frac{s_w}{\sigma} - g_n + \frac{\Delta s}{\sigma}(1-u)\varepsilon \right]$$

which is such that the equilibrium value of the employment rate is the same as in the original model

$$v^* = v^e = \frac{\alpha + \gamma}{\rho} \tag{23}$$

Then, an aid for choosing plausible values of the parameter to be used in the simulation is given by a very recent contribution by Harvie [26], where the author applies Goodwin's model to all the major OECD countries. Choosing, for example, his results for the UK economy [26, p. 362, Table 2], the parameter estimates are $\alpha = 0.0221$, $\beta = 0.0037$, $\sigma = 2.57$, $\gamma = 18.54$ and $\rho = 21.9$. From these values and condition (12) above, it also follows that the two propensities to save must satisfy

$$0 \le s_w < 0.066306 < s_c \le 1$$

Finally, after having chosen a value for the parameter δ $(= f_{\hat{v}})$ — for example $\delta = 0.1$ — and for the capitalists' propensity to save — for example

[14] In the version of the model we have considered, for example, we know that the results of the Hopf bifurcation are closed orbits of period approximately equal to $2\pi/\sqrt{B}$.

[15] For a discussion of the concept of "bifurcation" and of typical "routes to chaos" — the so-called *scenarios* — see, for example, Medio [33, pp. 59-69 and 149-177, respectively].

Table 1 Convergence to the steady-state for values of the workers' propensity to save less than the critical value

s_w	v^*	u^*	ε^*
0.020	0.847584	0.889490	0.722450
0.021	0.847584	0.889490	0.708069
0.022	0.847584	0.889490	0.693638
0.023	0.847584	0.889490	0.679158
0.024	0.847584	0.889490	0.664627
0.025	0.847584	0.889490	0.650045
0.026	0.847584	0.889490	0.635413
0.027	0.847584	0.889490	0.620730
0.028	0.847584	0.889490	0.605995
0.029	0.847584	0.889490	0.591209
0.0295	0.847584	0.889490	0.583796
0.02957	0.847584	0.889490	0.582758

$s_c = 0.6$ — it is possible to use (18) to calculate the critical value of the workers' propensity to save, for which the Hopf bifurcation occurs. Doing this, we obtain:

$$s_{wH} \approx 0.02957435 \tag{24}$$

Using these values for the parameters, we first notice that the numerical simulations[16] confirm the qualitative results we have obtained in the previous section; in particular:

- for values of s_w less than the critical value (24), the system converges to the steady-state, with the values for v^*, u^* and ε^* — given by (10), (11) and (23) — shown in Table 1. As was to be expected, only the steady-state value of ε proves to depend on s_w;[17]
- for $s_w = 0.029575$, the numerical simulation given in Figure 1 confirms the implications of **Proposition 1**, namely the existence of persistent and bounded fluctuations of the variables. Moreover, it gives strong evidence that the resulting closed orbit is stable (in other words, that the HB is super-critical).

However — as s_w is further increased — *new phenomena* appear. In particular, the bifurcation diagram given in Figure 2 suggests that, for higher values of the parameter, the original Hopf bifurcation is followed by other bifurcations on a route to *chaos*. To confirm this result we have calculated the dominant Lyapunov exponent of our dynamical system for different values of s_w. The resulting Lyapunov exponent bifurcation diagram given in Figure 3 shows that, for values of the workers' propensity to save greater than

[16] The numerical simulations have been performed using the computer program DYNAMICS 2, contained in the second edition of the book by Nusse and Yorke [34].

[17] From the results of the simulations, it also follows that the convergence to $(v^*, u^*, \varepsilon^*)$ is very slow (the slower, the larger is s_w) and *cyclical*.

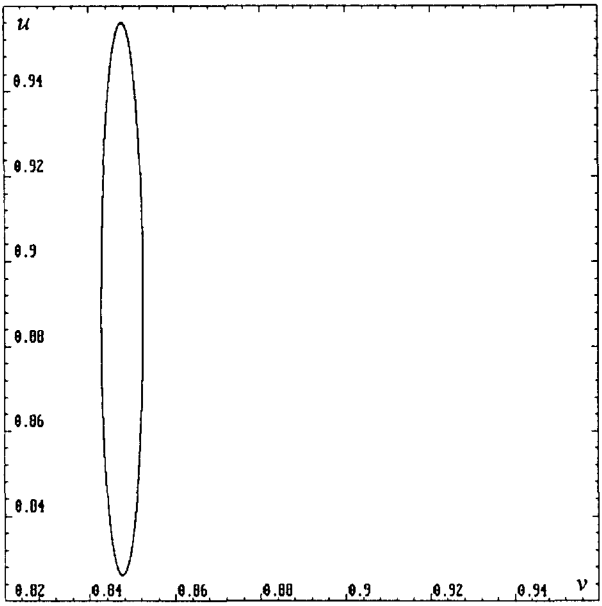

Fig. 1 Stable limit cycle in the (v, u)-plane, with $s_w = 0.029575$

a given critical value — approximately equal to 0.03 — the Lyapunov exponent is positive, and therefore the system possesses sensitive dependence on initial conditions.

4 Concluding remarks

In this paper, we have presented an extension of Goodwin's growth cycle model which, taking account of Pasinetti's criticism of Kaldor's theory of growth and income distribution, considers the case of differential savings. Notwithstanding its simplicity, the exercise we have performed allows us to draw some interesting and encouraging conclusions, worthy, in our opinion, of further investigation.

In summary:

1. The extension of the model we have considered attempts to integrate some Kaldorian-Pasinettian and Goodwinian elements. From this point of view, the result we have obtained is that the steady-state features of the modified Goodwin model are the same as those of Pasinetti's version of Kaldor's model. However, interestingly enough, along the equilibrium growth path of our model there is not full employment but, rather, a constant rate of unemployment. Moreover, for values of the workers' propensity to save above a certain critical value, the system — either periodically or aperiodically — persistently fluctuates around it, rather than converging to it.

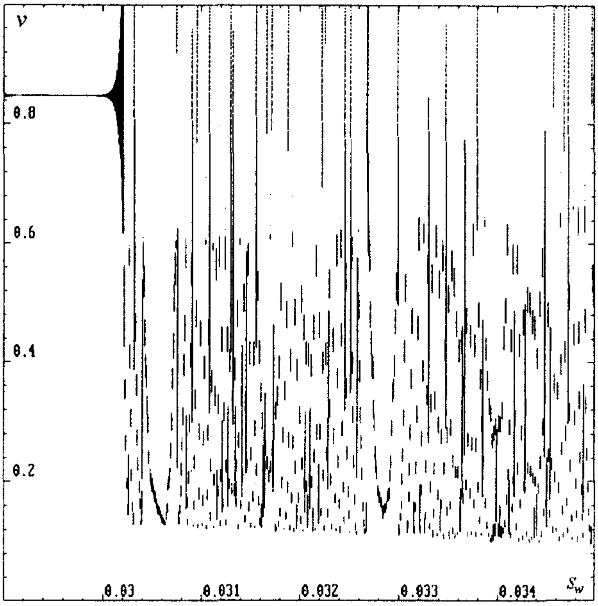

Fig. 2 Bifurcation diagram for the variable v, with the bifurcation parameter s_w in the range (0.029, 0.035).

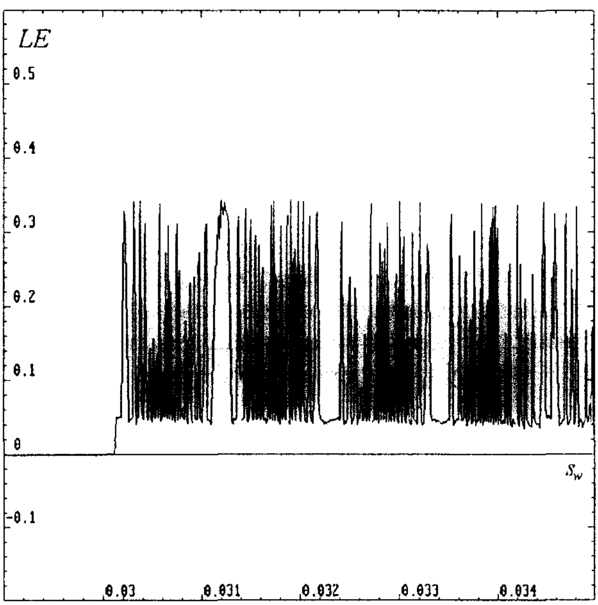

Fig. 3 Lyapunov exponent (LE) bifurcation diagram with the bifurcation parameter s_w in the range (0.029, 0.035)

2. The existence part of the HBT we have employed for the qualitative analysis of the dynamics of the model is a powerful tool, by now standard in economic dynamics. Yet, there are *three* aspects of our application that are worth stressing. *First*, as we have seen, our application does not require the introduction of any *ad hoc* assumption about the values of the parameters, apart from the one about the relative values of the two propensities to save implied by Kaldor-Pasinetti's theory: in our model this condition is all one needs to ensure both that the steady-state values of the variables are economically meaningful and that the conditions of the theorem are satisfied. *Second*, the results we have obtained strongly suggest the importance of "going from local to global analysis". From this point of view, the limit cycle, the existence of which we have proved by using the HBT, must be seen as the starting point rather than as the "goal" of the analysis. Indeed, more important, then, (as we have tried to do with the numerical simulation) is to investigate the possibility of other scenarios, for example the existence of a "route to chaotic behaviour", as the crucial parameter is further increased. *Third*, we have introduced the hypothesis of differential savings in the original formulation of the model. In doing this, we have neglected a number of interesting extensions of the model — available in the existing literature on the topic — which give rise to a higher order dynamical system, for example those extensions in which the Phillips curve is written in monetary terms and then an equation for price dynamics is introduced into the model or those extensions in which endogenous fluctuations in the capital-output ratio are allowed. Most importantly, concentrating on the Pasinettian side of the model, we have neglected the possibility of introducing into the model a separate investment function. A further elaboration of the model in this direction appears to be interesting and, to some extent, required if the purpose is properly to integrate Kaldorian and Goodwinian elements. However, this does not appear to be an impossible task. Indeed, as is testified to by some of the recent contributions cited above, the existence part of the HBT can be easily applied to higher than 3D systems as well, and the same, clearly, is true for the many numerical simulation techniques nowadays easily applicable with no more than a standard home PC. For this reason, then, there are no limits to the analysis of further generalizations of the model along the lines suggested in this paper.

A Appendix

Given (14), (15) and (16), we can express $AB - C$ as

$$\begin{aligned} AB - C &= (a_{22} + a_{33})(a_{23}a_{32} - a_{22}a_{33}) + a_{21}(a_{13}a_{32} + a_{22}a_{12}) \\ &= f_{\hat{v}}^{*}\varepsilon^{*}\frac{\Delta s}{\sigma}\left[f_{\hat{v}}^{*}u^{*} + (1-u^{*})\right]\left[\frac{\Delta s}{\sigma}u^{*}\frac{s_{w}}{\sigma}\varepsilon^{*} + \frac{\Delta s}{\sigma}\varepsilon^{*}u^{*}\frac{\Delta s}{\sigma}(1-u^{*})\varepsilon^{*}\right] \end{aligned}$$

$$+f_v^* u^* \frac{\Delta s}{\sigma} \varepsilon^* \left[v^* f_{\hat{v}}^* \frac{\Delta s}{\sigma} \varepsilon^* u^* - v^* \frac{s_w}{\sigma} \right]$$

Thus, given the expressions for ε^* and u^* in (10) and (11)

$$AB - C \gtreqless 0$$

according as to whether

$$\begin{aligned} & f_{\hat{v}}^* \left[f_{\hat{v}}^* u^* + (1 - u^*) \right] \left[\frac{\Delta s}{\sigma} \left(\frac{s_c - \sigma g_n}{s_c} \right) \left(\frac{s_w}{\sigma} \right) \frac{s_c (g_n - s_w/\sigma)}{\Delta s g_n} \right. \\ & + \left. \left(\frac{\Delta s}{\sigma} \right) \frac{s_c (g_n - s_w/\sigma)}{\Delta s g_n} \left(\frac{s_c - \sigma g_n}{s_c} \right) \left(\frac{\Delta s}{\sigma} \right) \left(\frac{\sigma g_n}{s_c} \right) \frac{s_c (g_n - s_w/\sigma)}{\Delta s g_n} \right] \\ & + f_v^* u^* \left[v^* f_{\hat{v}}^* \left(\frac{\Delta s}{\sigma} \right) \frac{s_c (g_n - s_w/\sigma)}{\Delta s g_n} \left(\frac{s_c - \sigma g_n}{s_c} \right) - v^* \frac{s_w}{\sigma} \right] \gtreqless 0 \end{aligned}$$

or

$$\begin{aligned} & f_{\hat{v}}^* \left[f_{\hat{v}}^* u^* + (1 - u^*) \right] \left(g_n - \frac{s_w}{\sigma} \right) \left(\frac{s_c}{\sigma} - g_n \right) \\ & + f_v^* u^* \left[v^* f_{\hat{v}}^* \frac{(g_n - s_w/\sigma)}{g_n} \left(\frac{s_c}{\sigma} - g_n \right) - v^* \frac{s_w}{\sigma} \right] \gtreqless 0 \end{aligned}$$

or

$$f_{\hat{v}}^* \left[f_{\hat{v}}^* u^* + (1 - u^*) \right] \left(g_n - \frac{s_w}{\sigma} \right) \frac{s_c}{\sigma} + f_v^* \left[v^* f_{\hat{v}}^* \left(1 - \frac{s_w}{\sigma g_n} \right) \frac{s_c}{\sigma} u^* - v^* \frac{s_w}{\sigma} \right] \gtreqless 0$$

or

$$\begin{aligned} & \left\{ -f_{\hat{v}}^* \left[f_{\hat{v}}^* u^* + (1 - u^*) \right] \frac{s_c}{\sigma} - f_v^* v^* f_{\hat{v}}^* \frac{s_c}{\sigma g_n} u^* - f_v^* v^* \right\} \frac{s_w}{\sigma} \\ & + f_{\hat{v}}^* \left[f_{\hat{v}}^* u^* + (1 - u^*) \right] \frac{g_n s_c}{\sigma} + f_v^* v^* f_{\hat{v}}^* \frac{s_c}{\sigma} u^* \gtreqless 0 \end{aligned}$$

or

$$\begin{aligned} & \left\{ \frac{f_{\hat{v}}^* s_c}{\sigma} \left[f_{\hat{v}}^* u^* + (1 - u^*) \right] + f_v^* v^* + f_v^* f_{\hat{v}}^* \left(\frac{u^* v^*}{1 - u^*} \right) \right\} s_w \\ & \lesseqgtr \sigma \left\{ \frac{f_{\hat{v}}^* s_c}{\sigma} \left[f_{\hat{v}}^* u^* + (1 - u^*) \right] + f_v^* f_{\hat{v}}^* \left(\frac{u^* v^*}{1 - u^*} \right) \right\} g_n \end{aligned}$$

from which (18) easily follows.

References

1. Toichiro Asada, "On a mixed competitive-monopolistic macrodynamic model in a monetary economy", Journal of Economics **54**, (1991) 33-53.
2. Anthony B. Atkinson, "The timescale of economic models: How long is the long run?", Review of Economic Studies **36**, (1969) 137-152.
3. Jess Benhabib and Takahiro Miyao, "Some new results on the dynamics of the generalized Tobin model", International Economic Review **36**, (1981) 589-598.
4. Jess Benhabib and Kazuo Nishimura, "The Hopf bifurcation and the existence and stability of closed orbits in multisectoral models of optimal economic growth", Journal of Economic Theory **21**, (1979) 421-444.
5. Jess Benhabib and Kazuo Nishimura, "Stability of equilibrium in dynamic models of capital theory", International Economic Review **22**, (1981) 275-293.
6. Franco Cugno and Luigi Montrucchio, "Cyclical growth and inflation: A qualitative approach to Goodwin's model with money prices", Economic Notes **11**, (1982) 93-107.
7. Meghnad Desai, "Growth cycles and inflation in a model of the class struggle", Journal of Economic Theory **6**, (1973) 527-545.
8. Luciano Fanti and Piero Manfredi, "Crescita con ciclo, ritardi nei piani di investimento ed effetti di popolazione", Rivista Internazionale di Scienze Sociali **105**, (1997) 79-114.
9. Luciano Fanti and Piero Manfredi, "Vischiosità dei salari e dinamica occupazionale in modelli di crescita alla Goodwin", Economia Politica **15**, (1998) 241-266.
10. Luciano Fanti and Piero Manfredi, "A Goodwin-type growth cycle model with profit-sharing", Economic Notes **27**, (1998) 371-402.
11. M. Farkas and M. Kotsis, "Modelling predator-prey and wage-employment dynamics", in Gustav Feichtinger (ed.), *Dynamic Economic Models and Optimal Control* (Springer-Verlag, Berlin 1992) 513-526.
12. Gustav Feichtinger, "Hopf bifurcation in an advertising diffusion model", Journal of Economic Behavior and Organization **17**, (1992) 401-412.
13. Piero Ferri and Edward Greenberg, *The Labor Market and Business Cycle Theories* (Springer-Verlag, Berlin 1989) 53-60.
14. Jean-Paul Fitoussi and Kumaraswamy Velupillai, "Growth and cycles in distributive shares of output and employment", in Jean-Paul Fitoussi and Pierre-Alain Muet (eds), *Macrodynamique et Déséquilibres* (Ed. Economica, Paris 1987) 5-54.
15. Peter Flaschel and Michael Krüger, "Endogenous aspirations in a model of cyclical growth", Ricerche Economiche **38**, (1984) 598-612.
16. Peter Flaschel and Gandolf Groh, "The classical growth cycle: Reformulation, simulation and some facts", Economic Notes **24**, (1995) 293-326.
17. Reiner Franke, "Stable, unstable, and persistent cyclical behavior in a Keynes-Wicksell monetary growth model", Oxford Economic Papers **44**, (1992) 242-256.
18. Reiner Franke and Toichiro Asada, "A Keynes-Goodwin model of the business cycle", Journal of Economic Behavior and Organization **24**, (1994) 273-295.
19. Michael Funke, "A generalized Goodwin model incorporating technical progress and variable prices", Economic Notes **16**, (1987) 36-47.
20. Giancarlo Gandolfo, *Economic Dynamics. Study Edition* (Springer-Verlag, Berlin 1997).

21. Jörg Glombowski and Michael Krüger, "On the role of distribution in different theories of cyclical growth", Research Memorandum, n. 132, Department of Economics, Tilburg University, Tilburg 1983.
22. Jörg Glombowski and Michael Krüger, "Some extensions of a classical growth cycle model", in Willi Semmler (ed.), *Competition, Instability, and Nonlinear Cycles* (Springer-Verlag, Berlin 1986) 212-251.
23. Jörg Glombowski and Michael Krüger, "Generalizations of Goodwin's growth cycle model", in David Batten, John Casti and Börje Johansson (eds), *Economic Evolution and Structural Adjustment* (Springer-Verlag, Berlin 1987) 260-290.
24. Jörg Glombowski and Michael Krüger, "A short-period growth cycle model", Recherches Econ. Louvain **54**, (1988) 423-438.
25. Richard Goodwin, "A Growth Cycle", in Charles H. Feinstein (ed.), *Socialism, Capitalism and Economic Growth* (Cambridge University Press, Cambridge 1967) 54-58.
26. David Harvie, "Testing Goodwin: growth cycles in ten OECD countries", Cambridge Journal of Economics **24**, (2000) 349-376.
27. Luigi Izzo, "La moneta in un modello di sviluppo ciclico", in *Saggi di analisi e teoria monetaria* (Franco Angeli Editore, Milano 1971) 184-210.
28. Nicholas Kaldor, "Alternative theories of distribution", Review of Economic Studies **23**, (1956) 83-100.
29. Christoph Kind, "Remarks on the economic interpretation of Hopf bifurcations", Economic Letters **62**, (1999) 147-154.
30. Hans-Walter Lorenz, *Nonlinear Economic Dynamics and Chaotic Motion*, 2nd edn (Springer-Verlag, Berlin 1993).
31. Alfredo Medio, *Teoria nonlineare del ciclo economico* (Il Mulino, Bologna 1979).
32. Alfredo Medio, "Oscillations in optimal growth models", Journal of Economic Behavior and Organization **8**, (1987) 413-427.
33. Alfredo Medio, *Chaotic Dynamics. Theory and Applications to Economics*, in collaboration with Giampaolo Gallo (Cambridge University Press, Cambridge 1992).
34. Helena E. Nusse and James A. Yorke, *Dynamics: Numerical Explorations*, 2nd, revised and enlarged edn (Springer-Verlag, Berlin 1998).
35. Luigi L. Pasinetti, "Rate of profit and income distribution in relation to the rate of economic growth", Review of Economic Studies, **29**, (1962) 267-279.
36. Luigi L. Pasinetti, "The rate of profit in an expanding economy", in *Growth and Income Distribution. Essays in Economic Theory* (Cambridge University Press, Cambridge 1974) 121-146.
37. Arthur W. Phillips, "The relation between unemployment and the rate of change of money wage rates in the United Kingdom, 1861-1957", Economica **25**, (1958) 283-299.
38. Frederick Ploeg, van der, "Economic growth and conflict over the distribution of income", Journal of Economic Dynamics and Control **6**, (1983) 253-279.
39. Frederick Ploeg, van der, "Predator-prey and neo-classical models of cyclical growth", Zeitschrift für Nationalökonomie **43**, (1983) 235-256.
40. Frederick Ploeg, van der, "Implications of workers' savings for economic growth and the class struggle", in Richard M. Goodwin, Michael Krüger and Alessandro Vercelli (eds), *Nonlinear Models of Fluctuating Growth* (Springer-Verlag, Berlin 1984) 1-13.

41. Frederick Ploeg, van der, "Growth cycles, induced technical change, and perpetual conflict over the distribution of income", Journal of Macroeconomics **9**, (1987) 1-12.
42. Kazuyuki Sasakura, "On the dynamic behavior of Schinasi's business cycle model", Journal of Macroeconomics **16**, (1994) 423-444.
43. Willi Semmler, "A macroeconomic limit cycle with financial perturbations", Journal of Economic Behavior and Organization **8**, (1987) 469-495.
44. Mario C. Sportelli, "A Kolmogoroff generalized predator-prey model of Goodwin's growth cycle", Journal of Economics **61**, (1995) 35-64.
45. Mario C. Sportelli and M. Prospero Cagnetta, *Le equazioni di Lotka-Volterra e la teoria economica del ciclo e della crescita* (Cacucci Editore, Bari 1990).
46. Steven H. Strogatz, *Nonlinear Dynamics and Chaos. With Applications to Physics, Biology, Chemistry and Engineering* (Addison-Wesley Publishing Company, Reading, Mass. 1994).
47. V. Torre, "Existence of limit cycles and control in complete Keynesian system by theory of bifurcations", Econometrica **45**, (1977) 1457-1466.
48. Kumaraswamy Velupillai, "Some stability properties of Goodwin's growth cycle", Zeitschrift für Nationalökonomie **39**, (1979) 245-257.
49. Kumaraswamy Velupillai, "Linear and nonlinear dynamics in economics: The contributions of Richard Goodwin", Economic Notes **11**, (1982) 73-92.