TASKS of MATHEMATICS
for economic applications AA. 2013/14

Intermediate Test December 2013

I M 1) Find all the complex numbers z such that $z^4 + i\,z = 0$.

I M 2) Given the linear system $\mathbb{A} \cdot \mathbb{X} = \mathbb{Y}$, with $\mathbb{A} = \begin{Vmatrix} 1 & -2 & 2 & -3 \\ 2 & 0 & 2 & 1 \\ 4 & 4 & 2 & 9 \end{Vmatrix}$, find all the vectors $\mathbb{Y}$ for which the system has solutions.

I M 3) The vector $\mathbb{X}$ has coordinates $(1, -2)$ in the base $\{(1,1);(2,1)\}$. Determine its coordinates (x_1, x_2) in the base $\{(2,1);(3,2)\}$.

I M 4) Given the matrix $\mathbb{A} = \begin{Vmatrix} 3 & -1 & k \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{Vmatrix}$, verify if, varying the parameter k, the matrix is a diagonalizable one when it admits multiple eigenvalues. And then, for which values of k the matrix admits complex eigenvalues ?

I M 5) Given a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3, f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$, find the matrix $\mathbb{A}$ and its eigenvalues knowing that:
.) the Kernel of the linear map consists in all the vectors orthogonal to the vector $(1, -1, 1)$;
..) the image of the vector $(1, 2, -1)$ is the vector $(4, 2, -2)$.

Intermediate Test January 2014

II M 1) Consider the function $f(x,y) = e^{x-y}$, the vector $v = (\cos\alpha, \sin\alpha)$ and the point $O = (0,0)$. Find, if they exist, angles α such that the second directional derivative $\mathcal{D}^2_{v,v} f(O) = 1$.

II M 2) Verify that the function $f(x,y) = \begin{cases} \dfrac{xy|xy|}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$ is differentiable at point $(0,0)$.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y,z) = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \\ \text{u.c. } x^2 + y^2 + z^2 = 3 \end{cases}$.

II M 4) Check that the condition $x\,e^{2y^2+z^2} = y\,e^{x^2+2z^2}$ can define in a neighbourhood of point $P(1,1,1)$ a function of two variables. Define the function and calculate the equation of its tangent plane at P.

II M 5) Consider the function $z = x^3 + \alpha xy + y^3$.
If $z''_{xx}(1,0) \cdot z''_{yy}(0,1) = z''_{xy}(1,0) \cdot z''_{yx}(0,1)$, wich values can assume α ?

I Winter Exam Session 2014

I M 1) Compute $\sqrt[4]{i^{68} - i^{57}}$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 2 & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & 2 \end{Vmatrix}$ find the value of the parameter k such that the matrix admits a multiple eigenvalue, and find, in such case, an orthogonal matrix which diagonalizes $\mathbb{A}$.

I M 3) Consider the matrix $\mathbb{A} = \begin{Vmatrix} 2 & 1 & \alpha \\ 1 & 1 & \beta \end{Vmatrix}$, $\alpha, \beta \in \mathbb{R}$ and $\mathbb{B} = \begin{Vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{Vmatrix}$. How many are the matrices $\mathbb{B}$ such that $\mathbb{A} \cdot \mathbb{B} = \mathbb{I}_2 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$?

I M 4) Given the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3, f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$, with $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & m \\ 1 & k & 1 \end{Vmatrix}$, check for the dimensions of the Kernel and of the Image, on varying the parameters m and k. Then find a basis for the Kernel and a basis for the Image of such linear map when the dimension of the Kernel is the maximum possible.

II M 1) Consider the two functions $F(x,y) = \left(x^2 + y^2, xy\right)$ and $g(t) = \left(\cos t, e^t\right)$. Calculate the derivative vector of composite $F(g(t))$ and find in parametric form the equation of its tangent line at $t = 0$.

II M 2) Determine the nature of quadratic form $q(x,y,z) = x^2 + 2y^2 - 2\alpha xy + \alpha z^2$ on varing the parameter α.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y,z) = x^2 + y^2 + z^2 \\ u.c. \ \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 3 \end{cases}$.

II M 4) Find the nature of point $x_0 = 0$ for function $y = y(x)$ defined in implicity form from the condition $(x+1)e^{x^2+y^2} = y^2\, e^{x+y}$ in a neighbourhood of the point $P(0,1)$.

II Winter Exam Session 2014

I M 1) Compute $\sqrt[3]{e^{\left(\log 2 + i\,\frac{\pi}{3}\right)}}$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 3 & 1 & 1 \\ 2 & k & 2 \\ 1 & 4 & 3 \end{Vmatrix}$ check if there are values of the parameter k such that the matrix is not a diagonalizable one.

I M 3) Determine all the linear maps $f : \mathbb{R}^3 \mapsto \mathbb{R}^2$ satisfying the following conditions:
a) they are surjective;
b) the vector $(1, 0, -1)$ belongs to the Kernel;
c) the image of the vector $(1, 2, 1)$ is the vector $(2, -4)$.

I M 4) The square matrix $\mathbb{A}_2$ admits the eigenvector $(1, 2)$ with the eigenvalue $\lambda = -1$ and the eigenvector $(2, -1)$ with the eigenvalue $\lambda = 1$. Determine the matrix $\mathbb{A}$ and its inverse matrix $\mathbb{A}^{-1}$. What do you suggest the result you have found ?

II M 1) Given $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} + \dfrac{x^2y^2}{x^2+y^2} & : (x,y) \neq (0,0) \\ k & : (x,y) = (0,0) \end{cases}$, study if exists a value of k such that the function is differentiable on $\mathbb{R}^2$.

II M 2) Given the function $f(x,y) = \log\left(x^2 - y^2\right)$, find point on $\mathbb{R}^2$ such that $\|\nabla f\|^2 = \det(\mathcal{H}f)$.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y) = x^2 + y \\ \text{u.c.} \begin{cases} x^2 + y^2 \le 1 \\ y - (x-1)^2 \ge 0 \end{cases} \end{cases}$.

II M 4) Check if with the set of conditions $\begin{cases} x^2 - y^2 + z^2 = 0 \\ xyz\, e^{xyz} = 0 \end{cases}$ we can define, in a neighbourhood of point $P(1,-1,0)$ a function in implicit form. If it is possible, define the function and calculate in parametric form the equation of its tangent line.

I Additional Exam Session 2014

I M 1) Compute $\sqrt[3]{(2+2i)^{10}}$ using trigonometric form for complex numbers.

I M 2) Given the matrix $\mathbb{A} = \left\| \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{matrix} \right\|$ find its eigenvalues and corresponding eigenvectors. The quadratic form $\mathbb{X} \cdot \mathbb{A} \cdot \mathbb{X}^{\mathrm{T}}$ is or not a definite one ?

I M 3) The linear map $F(x_1,x_2,x_3,x_4) : \mathbb{R}^4 \mapsto \mathbb{R}^3$:
$F(x_1,x_2,x_3,x_4) = (x_1+x_2+x_3+x_4, kx_1+x_2+x_3+mx_4, x_1+kx_2+mx_3+x_4)$
has an Image whose dimension is equal to 2.
The linear map $G(y_1,y_2,y_3) : \mathbb{R}^3 \mapsto \mathbb{R}^4$:
$G(y_1,y_2,y_3) = (y_1+y_3, ky_1+my_2+my_3, y_1+y_2+2y_3, ky_1+y_2+y_3)$
has a Kernel whose dimension is equal to 1.
Calculate the values of the parameters k and m and then find a basis for the Kernel of F and a basis for the Image of G.

I M 4) Find values for the parameter k such that the vectors $V=(k,k,1-k)$, $W=(1-k,1-k,1-k)$ and $Z=(1-k,k,k)$ are linearly dependent vectors.

II M 1) Given $f(x,y) = \begin{cases} \dfrac{x^2|y|^\alpha}{(x^2+y^2)^\alpha} & : (x,y) \ne (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$, where α is a positive real number, find values for α such that f is differentiable at $(0,0)$.

II M 2) The function $f(x,y) = \dfrac{1}{2} e^{2-(x^2+y^2)}$ has directional derivatives at point P; if $\mathcal{D}_v f(P) = 1$ and $\mathcal{D}_w f(P) = -1$, whit $v=(1,0)$ and $w=(0,1)$, find at least a point P that satisfies the proposed conditions and calculate the second directional derivative $\mathcal{D}^2_{v,w} f(P)$.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y) = 2x - 3y \\ \text{u.c.} \begin{cases} y \ge x^2 - x \\ y \le x \end{cases} \end{cases}$.

II M 4) Given the system $\begin{cases} f(x,y,z,w) = x^3y - y^2z^2 + zyw^2 = 1 \\ g(x,y,z,w) = e^x y - z\, e^w + x^2 zw = 1 \end{cases}$ and the point $\mathrm{P}_0 = (1,1,1,1)$ that satisfies it, verify that it is possible to define an implicit function $(x,y) \to (z(x,y), w(x,y))$ and then calculate the first order partial derivatives of such function.

I Summer Exam Session 2014

I M 1) Compute the square roots of the reciprocal of the number $1+\dfrac{1-2i}{2+i}$.

I M 2) Check for existence of solutions for the linear system $\begin{cases} 2x_1+18x_2-8x_3=20 \\ 3x_1-3x_2+6x_3=-6 \\ 4x_1+6x_2+kx_3=m \end{cases}$ on varying the parameters k and m.

I M 3) Find one matrix $\mathbb{B}=\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$ similar to the matrix $\mathbb{A}=\begin{Vmatrix} 1 & 2 \\ 4 & -1 \end{Vmatrix}$ and having as eigenvectors $(1,1)$ and $(1,0)$.

I M 4) Check for diagonalizability of the matrix $\mathbb{A}=\begin{Vmatrix} -7 & 2 & 5 \\ 2 & -2 & -2 \\ -5 & 2 & 3 \end{Vmatrix}$, finding its eigenvalues and also a basis for the corresponding eigenspace.

II M 1) The equation $e^{2x+3y}+2\,\text{sen}\,(x-y)-1=0$ defines at the point $P=(3\pi;\,-2\pi)$ only one between the two implicit functions $x=x(y)$ and $y=y(x)$. Check which function is defined and for this calculate the Taylor's polynomial of order two.

II M 2) The function $f(x,y)=(ax+by)\cdot\cos(x+y)$ has its gradient vector at point $(0,0)$ $\nabla f(0,0)$ equal to $(1,-2)$. Find the values of the parameters a and b and then check if the function $g(x,y)=\begin{cases} \dfrac{(f(x,y))^2}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ is differentiable at point $(0,0)$.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y)=xy+y \\ \text{u.c.} \begin{cases} y^2-2y-x\leq 0 \\ x-2y\leq 0 \end{cases} \end{cases}$.

II M 4) Given the functions:

$f:\mathbb{R}\to\mathbb{R}^2, f(t)=\left(t^2,1-t\right)$ and $g:\mathbb{R}^2\to\mathbb{R}^3, g(x,y)=\left(xy;e^{x+y};\dfrac{1}{x^2+y^2+1}\right)$,

determine the equation of the tangent line at point $t=0$ for the composite curve $g(f(t))$.

II Summer Exam Session 2014

I M 1) Compute $\sqrt{(1-i)^3}$.

I M 2) Determine the values of the parameter k for which the triangular matrix $\mathbb{A}=\begin{Vmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & k \end{Vmatrix}$ is a diagonalizable one.

I M 3) Given the matrix $\mathbb{A}=\begin{Vmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & m & k \end{Vmatrix}$, verify that it admits the eigenvalue $\lambda=2$ for any value of the parameters m and k, and then determine for which values of m and k it admits the eigenvalues $\lambda=5\pm i$.

I M 4) Given the matrix $\mathbb{A}=\begin{Vmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & m & k \\ 0 & k & 3 & -1 \end{Vmatrix}$, with m and k real parameters different from zero, consider the linear map $f:\mathbb{R}^4\to\mathbb{R}^3, f(\mathbb{X})=\mathbb{A}\cdot\mathbb{X}$. Determine, on varying m

and k, the dimension of the Kernel and of the Image of this linear map. When the dimension ot the Kernel is maximum, find a basis for it and a basis for the Image of the linear map.

II M 1) Given the system $\begin{cases} f(x,y,z) = (x+y+z)e^{x+y+z} = 0 \\ g(x,y,z) = xe^{yz} + ye^{xz} + ze^{xy} = 0 \end{cases}$, verify that it is not possible to define an implicit function $y \to (x,z)$ at the point $\mathrm{P}_0 = (1,0,-1)$ that satisfies it, whereas it is possible to define an implicit function $x \to (y,z)$. For this function determine the equation of the tangent line at the appropriate point.

II M 2) Check if the function $f(x,y) = x \cdot |\operatorname{sen} y|$ is differentiable at point $(0,0)$.

II M 3) Maximize the sum of three positive numbers x, y e z, under the condition that the sum of the square of the first number with twice the square of the second and three times the square of the third is equal to 11.

II M 4) Given the function $f(x,y) = x^3 + y^3$, u unit vector of $(1,-1)$ and v unit vector of $(1,1)$, find all the points (x,y) where it happens that $\mathcal{D}_u f(x,y) = 0$ and $\mathcal{D}_v f(x,y) = 3\sqrt{2}$.

I Autumn Exam Session 2014

I M 1) Compute $\sqrt[3]{1 + i^{10} - i^{15}}$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & k \end{Vmatrix}$, check if there are values of the parameter k such that the matrix admits multiple eigenvalues. Check if, for such values, the matrix is a diagonalizable one.

I M 3) Given the linear system $\begin{cases} x_1 + 2x_2 - x_4 = 2 \\ 2x_1 + x_2 - x_3 + m\,x_4 = k \\ -x_1 + 4x_2 + 2kx_3 + m\,x_4 = 4 \end{cases}$, verify that it always admits solutions whatever the values of the parameters m and k. Finally find the numerosity of such solutions on varying m and k.

I M 4) Determine the orthogonal matrix that diagonalizes $\mathbb{A} = \begin{Vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$.

II M 1) The equation $f(x,y,z) = xyz - e^{x-y} + e^{x-z} - e^{y-z} + 1 = 0$ defines at the point $P = (2,2,0)$ an implicit function $z = z(x,y)$. Calculate, for this function, the equation of the tangent plan at point $(2,2)$.

II M 2) The function $f(x,y) = ax^2 + by^3 + cx^2y^2$ has, at point $P = (-1,-1)$:

- partial derivative $f'_x(P) = -4$;
- directional derivative $\mathcal{D}_v f(P) = -3\sqrt{2}$;
- second order directional derivative $\mathcal{D}^2_{vv} f(P) = 7$.

Since $v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$, determine the values of the pararameters a, b, c.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y,z) = x^2 + y + z^2 \\ \text{u.c.: } x + y^2 + z = 0 \end{cases}$.

II M 4) Given the two functions $g : \mathbb{R}^2 \mapsto \mathbb{R}^3, g(t_1,t_2) = (x_1,x_2,x_3)$, differentiable at point P_0, and $f : \mathbb{R}^3 \mapsto \mathbb{R}^2, f(x_1,x_2,x_3) = (y_1,y_2)$, differentiable at point $g(P_0)$, knowing that

$\dfrac{\partial(x_1,x_2,x_3)}{\partial(t_1,t_2)}(P_0)=\begin{Vmatrix} 1 & 2 \\ 2 & -1 \\ -2 & 0 \end{Vmatrix}$ and $\dfrac{\partial(y_1,y_2)}{\partial(x_1,x_2,x_3)}(g(P_0))=\begin{Vmatrix} 2 & 2 & -1 \\ 1 & 0 & -1 \end{Vmatrix}$, calculate $\dfrac{\partial(y_1,y_2)}{\partial(t_1,t_2)}(P_0)$.

II Autumn Exam Session 2014

I M 1) Compute $\sqrt[4]{\dfrac{1-i}{1+i}}$.

I M 2) Given the matrices $\mathbb{A}=\begin{Vmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 2 & k & 0 \end{Vmatrix}$ and $\mathbb{B}=\begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & k \end{Vmatrix}$, check if there are values of the parameter k such that the matrices have the same determinant.

I M 3) Find the value of k such that the vectors $\mathbb{X}_1=(1,-1,3)$, $\mathbb{X}_2=(2,0,2)$ and $\mathbb{X}_3=(3,k,1)$ are linearly dependent vectors and then find the coefficients of their linear combination that gives as result the null vector.

I M 4) Find the eigenvalues for the matrix $\mathbb{A}=\begin{Vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{Vmatrix}$, and then check if such matrix is a diagonalizable one.

II M 1) The equation $f(x,y,z)=e^{x^2y^2z}-e^{xyz^2}=0$, satisfied at point $P=(1,1,0)$, defines an implicit function $z=z(x,y)$. Calculate the gradient vector of z at point $(1,1)$.

II M 2) Find maximum and minimum points for the function $f(x,y)=6xy-2x-3y$ in the triangle having vertices at points $(0,0)$, $(1,0)$ and $(0,1)$.

II M 3) Verify that the Hessian matrix of the function $f(x,y,z)=x^2\,y-x\,z^2-z^2+x^2$ can never be equal to the null matrix.

II M 4) Given $f(x,y)=x^2+y^2$, the unit vectors $u=(\cos\alpha,\operatorname{sen}\alpha)$, $v=(\cos\beta,\operatorname{sen}\beta)$, find the relation between α and β for which $\mathcal{D}^2_{u\,v}f(P)=0$.
Remember that $\cos\alpha\cos\beta+\operatorname{sen}\alpha\operatorname{sen}\beta=\cos(\alpha-\beta)$.

II Additional Exam Session 2014

I M 1) Given the matrix $\mathbb{A}=\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{Vmatrix}$, find its eigenvalues, write them in trigonometric form and then compute their product.

I M 2) Given the map $f:\mathbb{R}^3\mapsto\mathbb{R}^3$, $f(x_1,x_2,x_3)=(x_1-x_3,x_1+2x_2+x_3,kx_3-x_1)$, determine the value of the parameter k such that $\mathrm{Dim}(\mathrm{Ker}(f))=1$ and then find a basis for the Kernel and a basis for the Image of such linear map.

I M 3) Given the basis $\mathbb{W}=\{(1,0,1);(1,1,0);(0,1,1)\}$, find the representation of the vector $\mathbb{X}=(2,-1,2)$ under such basis.

I M 4) Find the matrix $\mathbb{B}=\begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}$ similar to the matrix $\mathbb{A}=\begin{Vmatrix} 1 & -1 \\ -2 & 2 \end{Vmatrix}$ if the similarity transformation is performed by the matrix $\mathbb{P}=\begin{Vmatrix} 1 & 1 \\ 2 & 3 \end{Vmatrix}$.

II M 1) The equation $f(x,y)=x^2+y^2+6xy+1=0$, satisfied at point $P=\left(\frac{1}{2},-\frac{1}{2}\right)$, defines an implicit function $y=y(x)$. Determine the expression of Taylor's second degree polynomial of such implicit function.

II M 2) Analyze the stationary points for the function $f(x,y)=y^2 e^x - xy$.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y,z)=x^2-y+z^2 \\ \text{u.c.: } x-y^2+z=0 \end{cases}$.

II M 4) Determine the values of the parameter k for which the quadratic form generated by the matrix $\mathbb{H}=\left\|\begin{matrix} k & 0 & 1 \\ 0 & k & 0 \\ 1 & 0 & k \end{matrix}\right\|$ is a positive or negative definite one.