TASKS of MATHEMATICS
for Economic Applications AA. 2014/15

Intermediate Test 13 December 2014

I M 1) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & k \\ 0 & -1 & 1 \end{Vmatrix}$ you must detect:

- the values for the parameter k for which the matrix is not diagonalizable;
- the value for the parameter k for which $\lambda = \cos\dfrac{\pi}{3} + i\,\text{sen}\,\dfrac{\pi}{3}$ is an eigenvalue of the matrix.

I M 2) Given the vectors $\mathbb{W}_1 = (1,1,0)$ and $\mathbb{W}_2 = (1,0,-1)$, consider the basis for $\mathbb{R}^3$: $\mathbb{W} = \{\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3\}$; knowing that the vector $\mathbb{X} = (3,-2,1)$ has coordinates $(2,1,2)$ in such basis, determine the vector $\mathbb{W}_3$.

I M 3) The matrix $\mathbb{A}$ has eigenvectors $\mathbb{X}_1 = (1,1,0)$, $\mathbb{X}_2 = (1,0,1)$ and $\mathbb{X}_3 = (0,1,1)$ corresponding to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = -1$. Determine the matrix $\mathbb{A}$, also using the similarity relation between matrices.

I M 4) Given the linear map $f : \mathbb{R}^3 \to \mathbb{R}^4, f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$, with $\mathbb{A} = \begin{Vmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & k \\ -1 & m & -1 \end{Vmatrix}$ determine the values of the parameters m and k so that the dimension of the Kernel is the maximum possible, and then find a basis for the Kernel and a basis for the Image of such linear map.

I M 5) Find an orthogonal matrix which diagonalizes $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{Vmatrix}$.

I Winter Exam Session 2015

I M 1) Find two numbers such that their sum is equal to 2 and their product is equal to 4. Then calculate their square roots.

I M 2) Consider a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3, f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$ such that:

a) $\mathbb{X}_1 = (1,1,1)$ is a basis for the Kernel;

b) $\mathbb{Y}_1 = (0,1,-1)$ is the image of $\mathbb{X}_2 = (1,1,0)$;

c) $\mathbb{Y}_2 = (2,-4,6)$ is the image of $\mathbb{X}_2 = (1,-1,1)$.

Find the matrix $\mathbb{A}$ of such linear map and then calculate its eigenvalues.

I M 3) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & k & 0 \\ 1 & 0 & 1 \end{Vmatrix}$ check for the values of the parameter k for which the matrix admits multiple eigenvalues. For such values find a basis for the corresponding eigenspaces of multiple eigenvalues.

I M 4) Given the linear system $\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ 3x_1 - x_2 + x_3 + 2x_4 = 2 \\ 3x_1 + m\,x_2 + 8x_3 + k\,x_4 = 1 \end{cases}$, check for existence of solutions on varying the parameters k and m.

II M 1) Check if the function $f(x,y) = |xy| \cdot (x - y)$ is differentiable at $(0,0)$.

II M 2) Given the function $f(x,y)=e^{x-y}$ and the two unit vectors $u=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ and $v=\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$, if $\mathcal{D}^2_{u,v}f(x,y)=-2$, find the value of $\mathcal{D}_u f(x,y)$.

II M 3) Check if with the system $\begin{cases} f(x,y,z)=xy-xz+y=0 \\ g(x,y,z)=xyz-xz+yz=0 \end{cases}$ we can define, in a neighbourhood of point $P(-1,1,0)$ a function in implicit form. If it is possible, define the function and calculate the equation of its tangent line.

II M 4) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y)=x^2-xy \\ \text{u.c.} \begin{cases} 1-x\le y \\ y\le 1-x^2 \end{cases} \end{cases}$.

II Winter Exam Session 2015

I M 1) Calculate the third order roots of the number $e^{\log 2+\frac{5}{3}\pi i}$.

I M 2) For the linear map $f:\mathbb{R}^4\to\mathbb{R}^3, f(\mathbb{X})=\mathbb{A}\cdot\mathbb{X}$ such that: $f(x_1,x_2,x_3,x_4)=(x_1+x_4,2x_1+x_2+x_3,x_1+x_2+x_3-x_4)$, find a basis for the Kernel and a basis for the Image of such linear map.

I M 3) Given the matrix $\mathbb{A}=\left\|\begin{matrix} 1 & 0 & k \\ 0 & 1 & 0 \\ k & 0 & 1 \end{matrix}\right\|$, since it admits the multiple eigenvalue $\lambda=1$, determine an orthogonal matrix that diagonalizes it.

I M 4) Given the matrix $\mathbb{A}=\left\|\begin{matrix} 3 & 0 & 5 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{matrix}\right\|$, determine its inverse matrix. With this, find the coordinates of the vector $(2,-1,3)$ in the basis $\mathbb{V}=\{(3,0,1);(0,1,0);(5,0,2)\}$.

II M 1) Given the function $f(x,y)=x^2-xy^2$ and the unit vectors u and v of the vectors $U=(1,1)$ and $V=(1,-1)$, find all the points (x,y) at which it is: $\begin{cases} \mathcal{D}_u f(x,y)=\sqrt{2} \\ \mathcal{D}^2_{u,v}f(x,y)=2 \end{cases}$.

II M 2) Check if with the equation $f(x,y)=y\cdot\log x-x\,e^y+x=0$, satisfied at the point $P=(1,0)$, it is possible to define an implicit function. If so, calculate its first and second order derivatives.

II M 3) Given the function $f(x,y)=x^2-xy^2+k\,xy$, analyze the nature of its stationary points on varying the parameter k.

II M 4) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y)=xy \\ \text{u.c.} \begin{cases} x^2-4x+4y\le 0 \\ x-4y\le 0 \end{cases} \end{cases}$.

I Additional Exam Session 2015

I M 1) Calculate the square roots of the number $z=5\left(\frac{1+i}{1+3i}-\frac{2-2i}{3+i}\right)$.

I M 2) For the linear map $f:\mathbb{R}^4\to\mathbb{R}^2, f(\mathbb{X})=\mathbb{A}\cdot\mathbb{X},\ \mathbb{A}=\left\|\begin{matrix} 1 & -1 & 2 & -2 \\ -2 & 2 & m & k \end{matrix}\right\|$, since $\text{Dim(Imm)}=1$, find the values of the parameters m and k and then find a basis for the Kernel and a basis for the Image of such linear map.

I M 3) Given the matrix $\mathbb{A} = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & k & 1 \end{Vmatrix}$, since it admits the eigenvalue $\lambda = -1$, find the value of the parameter k and then check if such matrix is a diagonalizable one.
I M 4) Check if there are values of x and y for which the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 2 \\ 1 & 3 \end{Vmatrix}$ is similar to the matrix $\mathbb{B} = \begin{Vmatrix} x & 1 \\ 1 & y \end{Vmatrix}$ if the matrix of the similarity transformation is $\mathbb{P} = \begin{Vmatrix} 2 & 1 \\ 3 & 2 \end{Vmatrix}$.
II M 1) The function $f(x,y) = e^{ax+y} + e^{x-by}$ has the following directional derivatives: $\mathcal{D}_v f(0,0) = 0$ and $\mathcal{D}^2_{v,-v} f(0,0) = 0$, where v is unit vector of $V = (1,-1)$. Find the values of the parameters a and b.
II M 2) The equation $f(x,y) = ye^{y-x} + e^{x-y} = 2$ defines at point $P = (1,1)$ an implicit function $y = y(x)$. Check for the nature of the point $x = 1$ for the function y.
II M 3) Consider the composite function $f \circ g = f(g(x,y))$ where:
$g : \mathbb{R}^2 \to \mathbb{R}^2,\ g(x,y) = (x-y, x+y) = (v,w)$,
$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(v,w) = \left(vw, v^2 - w^2\right)$.
Calculate the Jacobian matrix $J(f \circ g)$ of this composite function and then find the maximum possible value for the determinant $|J(f \circ g)| = |J(x,y)|$.
II M 4) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y) = x^2 + y \\ \text{u.c. } x^2 + y^2 = 1 \end{cases}$.

I Summer Exam Session 2015

I M 1) Calculate the roots of the equation $x^4 - 3x^2 - 4 = 0$, and then express such roots in trigonometric form.
I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & k \end{Vmatrix}$ check if there are values for the parameter k such that the matrix is a diagonalizable one.
I M 3) Given the linear system $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + x_3 = 1 \\ 4x_1 - 4x_2 + kx_3 = m \end{cases}$, check for existence of solutions on varying the parameters k and m.
I M 4) In the basis $\{(1,2,4);(2,0,-4);(-1,1,k)\}$ the vector $\mathbb{X} = (2,3,4)$ has coordinates $(1,1,1)$. Find the value of the parameter k.
II M 1) The equation $f(x,y) = xy - x^2 + y^2 - x^3 = 0$ defines at point $P = (-1,1)$ an implicit function $y = y(x)$. Compute $y'(-1)$ and $y''(-1)$, and then study the nature of the point $x = -1$.
II M 2) Given the function $f(x,y) = x^2 + y^2$ and the unit vectors $u = (\cos\alpha, \operatorname{sen}\alpha)$ and $v = (\cos\beta, \operatorname{sen}\beta)$, calculate $\mathcal{D}^2_{u,u} f(P_0), \forall P_0 \in \mathbb{R}^2$ and then find the relation between α and β if $\mathcal{D}^2_{u,v} f(P_0) = 2, \forall P_0 \in \mathbb{R}^2$ and if $\mathcal{D}^2_{u,v} f(P_0) = \sqrt{2}, \forall P_0 \in \mathbb{R}^2$.
II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y) = x - y \\ \text{u.c. } 4x^2 + y^2 \le 4 \end{cases}$.
II M 4) For the function $f(x,y) = xy^2 - x^2 - ky$, check for the nature of its stationary point on varying the parameter k.

II Summer Exam Session 2015

I M 1) Calculate $\sqrt[3]{i-1}$.

I M 2) Given the matrix $\mathbb{A}=\begin{Vmatrix}1 & 2 & -1\\ 2 & 0 & 1\\ 4 & -4 & k\end{Vmatrix}$, since it admits the eigenvalue $\lambda=0$, find the value of the parameter k and then check if such matrix is a diagonalizable one and find a basis for the eigenspace of its multiple eigenvalue.

I M 3) Given the linear map $f:\mathbb{R}^4\to\mathbb{R}^3, f(\mathbb{X})=\mathbb{A}\cdot\mathbb{X}$, with $\mathbb{A}=\begin{Vmatrix}1 & 1 & 1 & 1\\ 1 & 1 & m & 1\\ 1 & 1 & 1 & k\end{Vmatrix}$ determine the values of the parameters m and k so that the dimension of the Kernel is the maximum possible, and then find a basis for the Kernel and a basis for the Image of such linear map.

I M 4) Check if a matrix $\mathbb{B}$ may be similar to the matrix $\mathbb{A}=\begin{Vmatrix}1 & 2\\ 2 & -1\end{Vmatrix}$ when the matrix of the similarity transformation is $\mathbb{P}=\begin{Vmatrix}2 & 0\\ 1 & 2\end{Vmatrix}$. If it is possible, find $\mathbb{B}$.

II M 1) The equation $f(x,y)=x^2+y^2=1$ defines at point $P=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ an implicit function $y=y(x)$. Compute $y'\left(\frac{1}{\sqrt{2}}\right)$ and $y''\left(\frac{1}{\sqrt{2}}\right)$.

II M 2) Given the function $f(x,y)=x\,e^{x-y}$ and the unit vector $v=(\cos\alpha,\operatorname{sen}\alpha)$, since $\mathcal{D}_v f(1,1)=0$, calculate α.

II M 3) Solve the following optimization problem: $\begin{cases}\text{Max/min } f(x,y,z)=x-y+z\\ \text{u.c. } x^2+y^2+z^2=3\end{cases}$.

II M 4) Given the function $f(x,y)=x^2-4xy+y^2$, check for its maximum and minimum points in the domain $\mathcal{E}$, where $\mathcal{E}$ is the square having for vertices the points $(1,0)$, $(0,-1)$, $(-1,0)$ and $(0,1)$.

II Autumn Exam Session 2015

I M 1) After solving the equation $x^3-5x^2+4x-20=0$, compute the third roots of the solution having maximum modulus.

I M 2) Given the matrix $\mathbb{A}=\begin{Vmatrix}2 & 2 & 1\\ 0 & 1 & m\\ 1 & 2 & k\end{Vmatrix}$, since it admits the eigenvalue $\lambda=1$, find the values of the parameters m and k such that the matrix admits the multiple eigenvalue $\lambda=1$.

I M 3) Using the two vectors $\mathbb{X}_1=(1,0,1)$ and $\mathbb{X}_2=(0,1,0)$, determine a third vector $\mathbb{X}_3$ so as to get an orthonormal basis $\mathbb{X}=\{\mathbb{X}_1,\mathbb{X}_2,\mathbb{X}_3\}$ for $\mathbb{R}^3$. Given the vector $\mathbb{Y}$ with coordinates $(1,1,1)$ with respect to the canonical basis, find its coordinates in the basis $\mathbb{X}$.

I M 4) Determine at least one matrix that diagonalizes $\mathbb{A}=\begin{Vmatrix}0 & 0 & 2\\ 0 & 0 & 2\\ 2 & 2 & 0\end{Vmatrix}$.

II M 1) Given the function $f(x,y)=y^2\,e^x-y\,x^2$, find its maximum or minimum points.

II M 2) Given the quadratic form: $2(\mathrm{d}x)^2 + 2(\mathrm{d}y)^2 + k(\mathrm{d}z)^2 - 2\mathrm{d}x\mathrm{d}z + 2\mathrm{d}y\mathrm{d}z$, find the symmetric matrix that originates such quadratic form and then determine the values of the real parameter k for which such quadratic form is defined, semi-definite and indefinite.

II M 3) Solve the following optimization problem: $\begin{cases} \text{Max/min } f(x,y) = x + y \\ \text{u.c. } xy = k \end{cases}$ for $k > 0$ and then for $k < 0$.

II M 4) Given the system $\begin{cases} f(x,y,z) = e^{x^2-2y^2-1} - z = 0 \\ g(x,y,z) = z^3 + y^4 - x^2 = 0 \end{cases}$, check which type of implicit function is definable in a neighborhood of the point $P_0 = (1,0,1)$. Then determine, if it is possible, for such function the tangent vector at the appropriate point.