Prova intermedia di Analisi Matematica del 10/12/2013 — AM1

IM1) $z^5 = iz \Rightarrow z^5 - iz = 0 \Rightarrow z(z^4 - i) = 0 \Rightarrow z = 0$ e $z^4 = i \Rightarrow z = \sqrt[4]{i}$.

$i = 1\cdot\left(\cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2}\right) \Rightarrow \sqrt[4]{i} = \sqrt[4]{1}\cdot\left(\cos\left(\frac{\pi}{8} + k\cdot\frac{2\pi}{4}\right) + i\operatorname{sen}\left(\frac{\pi}{8}+k\cdot\frac{2\pi}{4}\right)\right);\ 0 \le k \le 3.$

$k=0:\ \cos\frac{\pi}{8} + i\operatorname{sen}\frac{\pi}{8}$; $k=1:\ \cos\frac{5}{8}\pi + i\operatorname{sen}\frac{5}{8}\pi$;

$k=2:\ \cos\frac{9}{8}\pi + i\operatorname{sen}\frac{9}{8}\pi$; $k=3:\ \cos\frac{13}{8}\pi + i\operatorname{sen}\frac{13}{8}\pi$.

IM2) $\sum_{n=0}^{+\infty} 2^n\cdot\log^{n+1}x = \sum_{n=0}^{+\infty}\log x\cdot(2\log x)^n = \log x\cdot\sum_{n=0}^{+\infty}(2\log x)^n.$

Studiamo la serie come serie geometrica di ragione $q = 2\log x$.

Per la convergenza dovrà essere: $|2\log x| < 1 \Rightarrow -1 < 2\log x < 1 \Rightarrow$

$\Rightarrow -\frac{1}{2} < \log x < \frac{1}{2} \Rightarrow e^{-\frac{1}{2}} < x < e^{\frac{1}{2}} \Rightarrow \frac{1}{\sqrt{e}} < x < \sqrt{2}.$

Se $x = \frac{1}{\sqrt{e}}:\ \sum_{n=0}^{+\infty}\left(2\cdot\log\frac{1}{\sqrt{e}}\right)^n = \sum_{n=0}^{+\infty} 2^n\cdot(-1)^n\cdot\frac{1}{2^n} = \sum_{n=0}^{+\infty}(-1)^n$: Serie indeterminata.

Se $x = \sqrt{e}:\ \sum_{n=0}^{+\infty}(2\cdot\log\sqrt{e})^n = \sum_{n=0}^{+\infty}2^n\cdot\frac{1}{2^n} = \sum_{n=0}^{+\infty}1$: Serie divergente.

Quindi $C = \left]\frac{1}{\sqrt{e}};\sqrt{e}\right[$. Funzione Somma: $S(x) = \log x\cdot\frac{1}{1-2\log x} = \frac{\log x}{1-2\log x}$.

IM3) $\underset{(t_1;t_2)}{\mathbb{R}^2} \xrightarrow{g} \underset{(x_1;x_2;x_3)}{\mathbb{R}^3} \xrightarrow{f} \underset{y}{\mathbb{R}}$. $\frac{\partial(y)}{\partial(t_1;t_2)} = \frac{\partial(y)}{\partial(x_1;x_2;x_3)}\cdot\frac{\partial(x_1;x_2;x_3)}{\partial(t_1;t_2)} \Rightarrow$

$$\Rightarrow \left\|\frac{\partial y}{\partial t_1}\ \ \frac{\partial y}{\partial t_2}\right\| = \left\|\frac{\partial y}{\partial x_1}\ \ \frac{\partial y}{\partial x_2}\ \ \frac{\partial y}{\partial x_3}\right\|\cdot\left\|\begin{matrix}\frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2}\\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2}\\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2}\end{matrix}\right\| \Rightarrow$$

$$\Rightarrow \left\|\frac{\partial y}{\partial t_1}\ \ \frac{\partial y}{\partial t_2}\right\| = \left\|e^{x_2-x_3}\ \ x_1e^{x_2-x_3}\ \ -x_1e^{x_2-x_3}\right\|\cdot\left\|\begin{matrix}3 & 2\\ t_2 & t_1\\ 1 & -2\end{matrix}\right\| \Rightarrow$$

$$\Rightarrow\begin{cases}\frac{\partial y}{\partial t_1} = 3e^{x_2-x_3} + t_2x_1e^{x_2-x_3} - x_1e^{x_2-x_3} = e^{x_2-x_3}\cdot(3 + t_2x_1 - x_1)\\ \frac{\partial y}{\partial t_2} = 2e^{x_2-x_3} + t_1x_1e^{x_2-x_3} + 2x_1e^{x_2-x_3} = e^{x_2-x_3}\cdot(2 + t_1x_1 + 2x_1)\end{cases} \Rightarrow$$

$$\Rightarrow\begin{cases}\frac{\partial y}{\partial t_1} = e^{t_1t_2 - t_1 + 2t_2}\cdot(3 + 3t_1t_2 + 2t_2^2 - 3t_1 - 2t_2)\\ \frac{\partial y}{\partial t_2} = e^{t_1t_2 - t_1 + 2t_2}\cdot(2 + 3t_1^2 + 2t_1t_2 + 6t_1 + 4t_2)\end{cases}.$$

IM4) $f(x;y) = xy + e^{x-y} - 2x + \log y = 0$; $f(1;1) = 1 + 1 - 2 + 0 = 0$.

$\nabla f(x;y) = \left(y + e^{x-y} - 2;\ x - e^{x-y} + \frac{1}{y}\right)$; $\nabla f(1;1) = (1+1-2;\ 1-1+1) = (0;1)$.

È possibile definire $y = y(x)$ con $y'(1) = -\frac{0}{1} = 0$: 1 è punto stazionario.

AM2

$$H=\begin{Vmatrix} e^{x-y} & 1-e^{x-y} \\ 1-e^{x-y} & e^{x-y}-\frac{1}{y^2} \end{Vmatrix}; \quad H(1;1)=\begin{Vmatrix} 1 & 1-1 \\ 1-1 & 1-1 \end{Vmatrix}=\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}.$$

$$y''(1) = -\frac{1+2\cdot0\cdot0+0\cdot0}{1} = -1<0: \quad x=1 \text{ è punto di massimo relativo.}$$

IM5) $f(x;y)=e^{x-y}$: funzione differenziabile due volte $\forall (x;y)\in\mathbb{R}^2$

$\nabla f(x;y)=\left(e^{x-y}; -e^{x-y}\right)$. $\nabla f(0;0)=(1;-1)$. $v=(\cos\alpha;\operatorname{sen}\alpha)$.

$\mathcal{D}_v f(0;0)=\nabla f(0;0)\cdot(\cos\alpha;\operatorname{sen}\alpha)=(1;-1)\cdot(\cos\alpha;\operatorname{sen}\alpha)=\cos\alpha-\operatorname{sen}\alpha.$

$$H(f)=\begin{Vmatrix} e^{x-y} & -e^{x-y} \\ -e^{x-y} & e^{x-y} \end{Vmatrix}; \quad H(0;0)=\begin{Vmatrix} 1 & -1 \\ -1 & 1 \end{Vmatrix}.$$

$$\mathcal{D}^2_{v,v} f(0;0) = v\cdot H\cdot v^T = \begin{Vmatrix} \cos\alpha & \operatorname{sen}\alpha \end{Vmatrix}\cdot\begin{Vmatrix} 1 & -1 \\ -1 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} \cos\alpha \\ \operatorname{sen}\alpha \end{Vmatrix} =$$

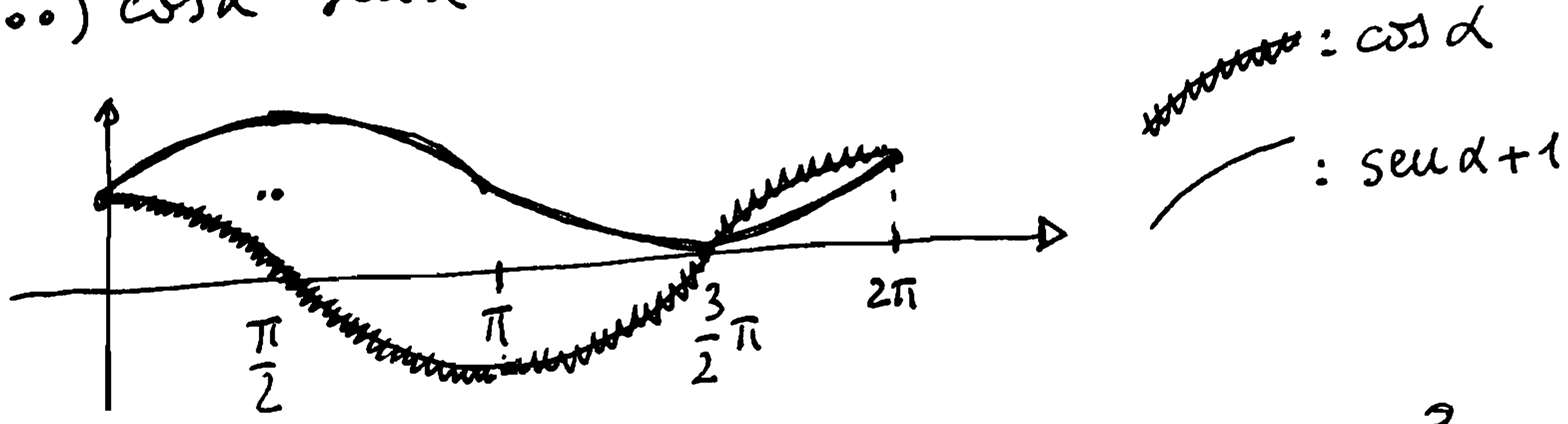
$$=\cos^2\alpha-2\operatorname{sen}\alpha\cos\alpha+\operatorname{sen}^2\alpha=(\cos\alpha-\operatorname{sen}\alpha)^2.$$

$\mathcal{D}_v f(0;0)=\mathcal{D}^2_{v,v} f(0;0) \Rightarrow \cos\alpha-\operatorname{sen}\alpha=(\cos\alpha-\operatorname{sen}\alpha)^2 \Rightarrow$

$\Rightarrow (\cos\alpha-\operatorname{sen}\alpha)^2-(\cos\alpha-\operatorname{sen}\alpha)=(\cos\alpha-\operatorname{sen}\alpha)\cdot(\cos\alpha-\operatorname{sen}\alpha-1)=0.$

$\cdot$) $\cos\alpha-\operatorname{sen}\alpha=0 \Rightarrow \cos\alpha=\operatorname{sen}\alpha \Rightarrow \alpha=\frac{\pi}{4}$ e $\alpha=\frac{5}{4}\pi$.

$\cdot\cdot$) $\cos\alpha-\operatorname{sen}\alpha-1=0 \Rightarrow \cos\alpha=\operatorname{sen}\alpha+1$. Graficamente:



Quindi $\cos\alpha=\operatorname{sen}\alpha+1$ se $\alpha=0$ e $\alpha=\frac{3}{2}\pi$.