Compito di Analisi Matematica del 17/2/14 [AM1]

IM1) $\sqrt[3]{e^{\log 2+i\frac{\pi}{3}}}=\sqrt[3]{e^{\log 2}\cdot e^{i\frac{\pi}{3}}}=\sqrt[3]{2\left(\cos\frac{\pi}{3}+i\,\mathrm{sen}\frac{\pi}{3}\right)}=$

$=\sqrt[3]{2}\cdot\left(\cos\left(\frac{\pi}{9}+k\cdot\frac{2\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{9}+k\cdot\frac{2\pi}{3}\right)\right);\ 0\le k\le 2.$

$k=0:\ \sqrt[3]{2}\left(\cos\frac{\pi}{9}+i\,\mathrm{sen}\frac{\pi}{9}\right);\ k=1:\ \sqrt[3]{2}\cdot\left(\cos\frac{7}{9}\pi+i\,\mathrm{sen}\frac{7}{9}\pi\right);\ k=2:\ \sqrt[3]{2}\left(\cos\frac{13}{9}\pi+i\,\mathrm{sen}\frac{13}{9}\pi\right).$

IM2) $f(x;y)=\begin{cases}\dfrac{x^2y^\alpha}{x^2+y^2} & :(x;y)\neq(0;0)\\ 0 & :(x;y)=(0;0)\end{cases}.\qquad \begin{cases}x=\rho\cos\vartheta\\ y=\rho\,\mathrm{sen}\,\vartheta\end{cases}$

$$\lim_{(x;y)\to(0;0)}\frac{x^2y^\alpha}{x^2+y^2}\Rightarrow\lim_{\rho\to 0}\frac{\rho^{2+\alpha}\cos^2\vartheta\,\mathrm{sen}^\alpha\vartheta}{\rho}=\lim_{\rho\to 0}\rho^{1+\alpha}(\cos^2\vartheta\cdot\mathrm{sen}^\alpha\vartheta)=0$$

sicuramente in modo uniforme se $\alpha>0$.

$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{h^2\cdot 0}{h^2}-0\right)=\lim_{h\to 0}0=0;$

$\frac{\partial f}{\partial y}(0;0)=\lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{0\cdot h^\alpha}{h^2}-0\right)=\lim_{h\to 0}0=0.$

$$\lim_{(x;y)\to(0;0)}\frac{\frac{x^2y^\alpha}{x^2+y^2}-0-(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}\Rightarrow\lim_{\rho\to 0}\frac{\rho^{2+\alpha}\cdot\cos^2\vartheta\cdot\mathrm{sen}^\alpha\vartheta}{\rho^3}=0$$

con convergenza uniforme se $2+\alpha>3\Rightarrow\alpha>1$.
La funzione è differenziabile se $\alpha>1$.

IM3) $\begin{cases}f(x;y;z)=x^2-y^2+z^2=0\\ g(x;y;z)=xyz\cdot e^{xyz}=0\end{cases};\ P_0=(1;-1;0);\ \begin{cases}f(P_0)=0\\ g(P_0)=0\end{cases}.$

$$\frac{\partial(f;g)}{\partial(x;y;z)}=\left\|\begin{matrix}2x & -2y & 2z\\ yze^{xyz}+xy^2z^2e^{xyz} & xze^{xyz}+x^2yz^2e^{xyz} & xye^{xyz}+x^2y^2ze^{xyz}\end{matrix}\right\|$$

$\frac{\partial(f;g)}{\partial(x;y;z)}(1;-1;0)=\left\|\begin{matrix}2 & 2 & 0\\ 0 & 0 & -1\end{matrix}\right\|$. Dato che $\begin{vmatrix}2 & 0\\ 0 & -1\end{vmatrix}=-2\neq 0$ si può definire una funzione implicita $x\to(y(x);z(x))$.

$\frac{dy}{dx}(1)=-\dfrac{\begin{vmatrix}2 & 0\\ 0 & -1\end{vmatrix}}{\begin{vmatrix}2 & 0\\ 0 & -1\end{vmatrix}}=-\frac{-2}{-2}=-1;\ \frac{dz}{dx}(1)=-\dfrac{\begin{vmatrix}2 & 2\\ 0 & 0\end{vmatrix}}{\begin{vmatrix}2 & 0\\ 0 & -1\end{vmatrix}}=-\frac{0}{-2}=0.$

Equazione retta tangente: $x\to(-1;0)+x(-1;0);\ x\to(-1-x;0)$.

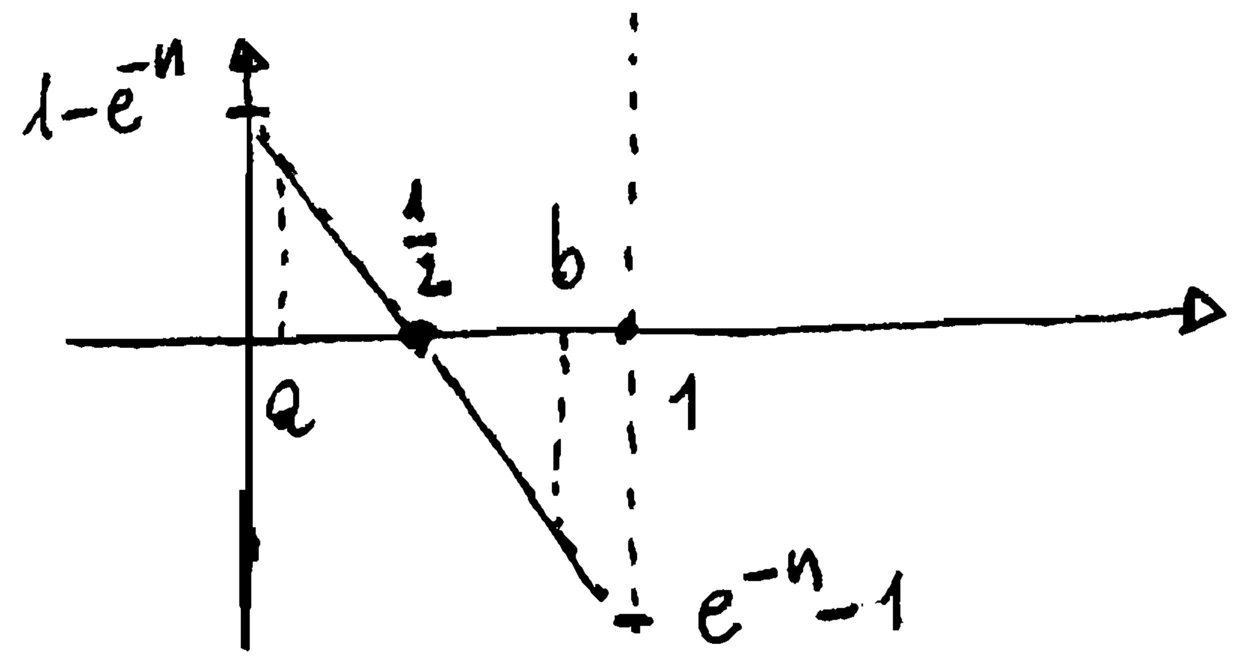
IM4) $f_n(x)=e^{-nx}-e^{n(x-1)}$. $\lim_{n\to+\infty}f_n(x)=\begin{cases}x<0: & +\infty\\ x=0: & f_n(0)=1-e^{-n}\to 1\\ 0<x<1: & 0\\ x=1: & f_n(1)=e^{-n}-1\to -1\\ x>1: & -\infty\end{cases}.$

Quindi $\mathcal{C} = [0;1]$ con $f(x) = \begin{cases} 1 : x = 0 \\ 0 : 0 < x < 1 \\ -1 : x = 1 \end{cases}$.

AM2

$f_n(0) = 1 - e^{-n}$; $f_n(1) = e^{-n} - 1$.

$f_n'(x) = -n \cdot e^{-nx} - n e^{n(x-1)} = -n(e^{-nx} + e^{n(x-1)}) < 0 \quad \forall x \in \mathbb{R}$.



$1 - e^{-n} \to 1$ per $n \to +\infty$

$e^{-n} - 1 \to -1$ per $n \to +\infty$

$e^{-nx} - e^{n(x-1)} = 0 \Rightarrow e^{-nx} = e^{n(x-1)} \Rightarrow$

$\Rightarrow -x = x - 1 \Rightarrow x = \frac{1}{2}$. $f_n(\frac{1}{2}) = 0 \ \forall n$.

Preso $[a;b] \subset]0;1[$ risulta $\sup_{x \in [a;b]} \{|f_n(x) - f(x)|\} = f(a)$ oppure $f(b)$

ma comunque $\lim_{n \to +\infty} f(a) = \lim_{n \to +\infty} f(b) = 0$ e quindi la

convergenza è uniforme in $[a;b]$.

$\sum_{n:0}^{+\infty} e^{-nx} - e^{n(x-1)} = \sum_{n:0}^{+\infty} e^{-nx} - \sum_{n:0}^{+\infty} e^{n(x-1)} = \sum_{n:0}^{+\infty} (e^{-x})^n - \sum_{n:0}^{+\infty} (e^{x-1})^n$.

Come serie geometriche dovrà essere, per la convergenza:

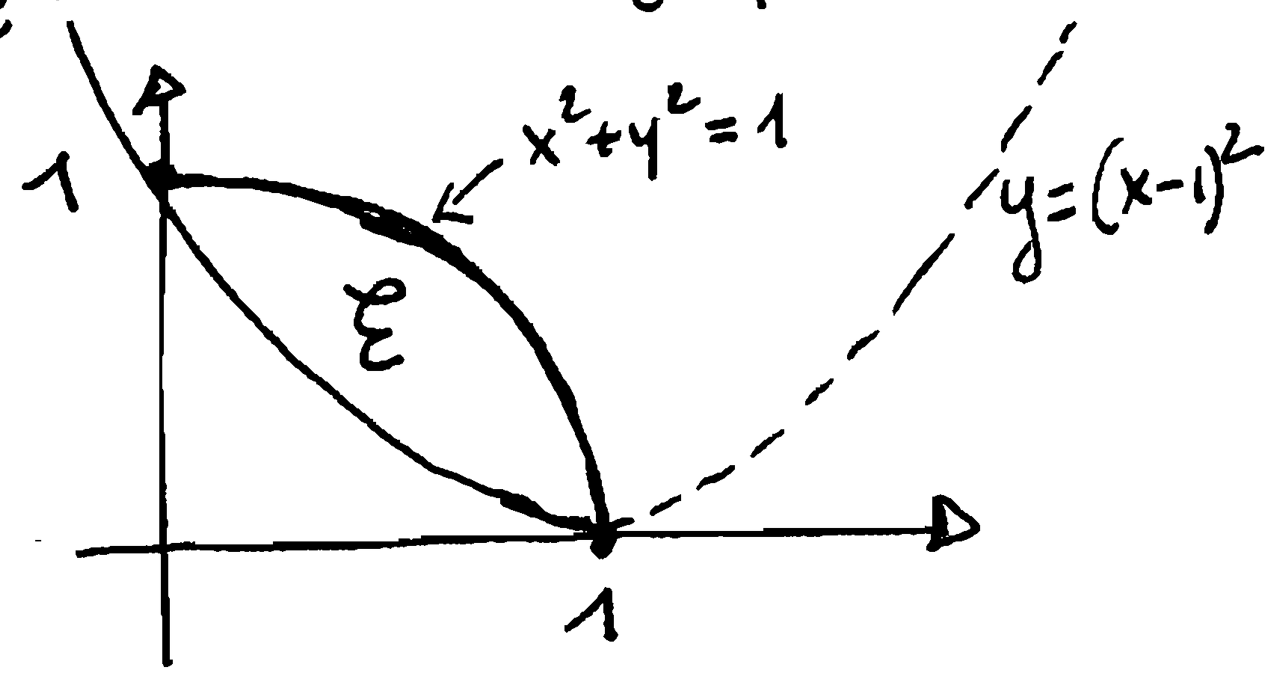
•) $|e^{-x}| < 1 \Rightarrow 0 < e^{-x} < 1 \Rightarrow x > 0$

••) $|e^{x-1}| < 1 \Rightarrow 0 < e^{x-1} < 1 \Rightarrow x - 1 < 0 \Rightarrow x < 1$

$\Rightarrow$ Converge in $]0;1[$.

Funzione Somma: $\sum_{n:0}^{+\infty} (e^{-x})^n - \sum_{n:0}^{+\infty} (e^{x-1})^n = \frac{1}{1 - e^{-x}} - \frac{1}{1 - e^{x-1}} = \frac{e^x}{e^x - 1} - \frac{e}{e - e^x}$; $0 < x < 1$.

II M1) $\begin{cases} \text{Max/min } f(x;y) = x^2 + y \\ \text{s.v. } \begin{cases} x^2 + y^2 \leq 1 \\ y - (x-1)^2 \geq 0 \end{cases} \end{cases}$



$\begin{cases} x^2 + y^2 = 1 \\ y = (x-1)^2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$ e $\begin{cases} x = 0 \\ y = 1 \end{cases}$.

$\mathcal{E}$ insieme limitato e chiuso, $f(x;y)$ continua, vincoli qualificati.

$\Lambda = x^2 + y - \lambda_1(x^2 + y^2 - 1) - \lambda_2((x-1)^2 - y)$

<u>Caso $\lambda_1 = \lambda_2 = 0$</u>

$\begin{cases} \Lambda'_x = 2x = 0 \Rightarrow x = 0 \\ \Lambda'_y = 1 \neq 0 \ \forall (x;y) \end{cases}$: non ci sono soluzioni.

Caso $\lambda_1 \neq 0;\ \lambda_2 = 0$ $\boxed{AM3}$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 2x(1-\lambda_1) = 0 \\ \Lambda'_y = 1 - 2\lambda_1 y = 0 \\ x^2+y^2 = 1 \\ y \geq (x-1)^2 \end{cases} \Rightarrow \begin{cases} x=0 \\ \lambda_1 = \frac{1}{2y} \\ y^2=1 \\ y \geq (x-1)^2 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ \lambda_1 = \frac{1}{2} \\ 1 \geq 1 : \text{vera} \end{cases} \text{ e } \begin{cases} x=0 \\ y=-1 \\ \lambda_1 = -\frac{1}{2} \\ -1 \geq 1 : \text{falsa.} \end{cases}$$

$\underline{\text{Max}}$?

$$\begin{cases} \lambda_1 = 1 \\ y = \frac{1}{2\lambda_1} = \frac{1}{2} \\ x^2 = 1 - \frac{1}{4} = \frac{3}{4} \\ y \geq (x-1)^2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 1 \\ x = \frac{\sqrt{3}}{2} \\ y = \frac{1}{2} \\ \frac{1}{2} \geq \frac{7}{4} - \sqrt{3} : \text{vera} \end{cases} \text{ e } \begin{cases} \lambda_1 = 1 \\ x = -\frac{\sqrt{3}}{2} \\ y = \frac{1}{2} \\ \frac{1}{2} \geq \frac{7}{4} + \sqrt{3} : \text{falsa.} \end{cases}$$

$\underline{\text{Max}}$?

Caso $\lambda_1 = 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_2(x-1) = 0 \\ \Lambda'_y = 1 + \lambda_2 = 0 \\ y = (x-1)^2 \\ x^2+y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} \lambda_2 = -1 \\ 2x+2x-2=0 \Rightarrow x = \frac{1}{2} \\ y = \frac{1}{4} \\ \frac{1}{4} + \frac{1}{16} \leq 1 : \text{vera} \end{cases} : \underline{\text{Min}}?$$

Caso $\lambda_1 \neq 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x - 2\lambda_2(x-1) = 0 \\ \Lambda'_y = 1 - 2\lambda_1 y + \lambda_2 = 0 \\ x^2+y^2=1 \\ y = (x-1)^2 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \end{cases} \text{ e } \begin{cases} x=1 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ 2\lambda_2 = 0 \\ 1 - 2\lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ \lambda_2 = 0 \\ \lambda_1 = \frac{1}{2} \end{cases} \text{ già visto } \underline{\text{Max}}?$$

$$\text{e } \begin{cases} x=1 \\ y=0 \\ 2-2\lambda_1 = 0 \\ 1+\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=0 \\ \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases} : \textit{Nulla.}$$

Sul vincolo $x^2+y^2=1$ risulta $\bar{H}(x;y;\lambda_1) = \left\| \begin{matrix} 0 & 2x & 2y \\ 2x & 2-2\lambda_1 & 0 \\ 2y & 0 & -2\lambda_1 \end{matrix} \right\| \Rightarrow$

$$\Rightarrow \left|\bar{H}\left(0;1;\tfrac{1}{2}\right)\right| = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = 2(0-2) = -4 < 0 \Rightarrow \text{punto di minimo.}$$

L'indicazione del $\lambda_1 = \frac{1}{2} > 0$ contraddice quella di $|\bar{H}|$ e quindi il punto $(0;1)$ non è né di massimo né di minimo.

Rimangono solo $\left(\frac{\sqrt{3}}{2};\frac{1}{2}\right)$ e $\left(\frac{1}{2};\frac{1}{4}\right)$ che, per il Teorema di Weierstrass, sono il punto di Massimo e quello di minimo (assoluti).

$\boxed{A014}$

II M2) $\begin{cases} x' = x - y + t \\ y' = y + 2 \end{cases}$. Si potrebbe risolvere $y' - y = 2$ e sostituire nella prima. Usiamo il metodo tradizionale.

$$\begin{cases} x' - x + y = t \\ 0 + y' - y = 2 \end{cases} \Rightarrow \begin{Vmatrix} D-1 & 1 \\ 0 & D-1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} t \\ 2 \end{Vmatrix} \Rightarrow \begin{vmatrix} D-1 & 1 \\ 0 & D-1 \end{vmatrix} (x) = \begin{vmatrix} t & 1 \\ 2 & D-1 \end{vmatrix} \Rightarrow$$

$$\Rightarrow (D-1)^2 (x) = (D-1)(t) - 2 = 1 - t - 2 = -t - 1 = x'' - 2x' + x.$$

$(\lambda - 1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1 \Rightarrow x(t) = c_1 e^t + c_2 t e^t$: Soluzione generale omogenea.

Soluzione particolare: $x_0(t) = at + b \Rightarrow x_0'(t) = a$; $x_0''(t) = 0$.

Sostituendo in $x'' - 2x' + x \Rightarrow 0 - 2a + at + b = -t - 1 \Rightarrow$

$$\Rightarrow \begin{cases} at = -t \\ -2a + b = -1 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = -1 - 2 = -3 \end{cases} \Rightarrow x_0(t) = -t - 3.$$

Soluzione generale per $x(t)$: $x(t) = c_1 e^t + c_2 t e^t - t - 3$.

Dalla prima equazione: $y = x - x' + t \Rightarrow$

$$\Rightarrow y(t) = \cancel{c_1} e^t + \cancel{c_2 t e^t} \cancel{-t} - 3 \cancel{- c_1 e^t} - c_2 e^t - \cancel{c_2 t} e^t + 1 \cancel{+ t} \Rightarrow$$

$$\Rightarrow y(t) = -c_2 e^t - 2.$$

II M3) $\begin{cases} y' - y = x y^2 \\ y(0) = 1 \end{cases}$. Equazione del tipo Bernoulli.

Posto $y^{1-2} = y^{-1} = \dfrac{1}{y} = w \Rightarrow y = \dfrac{1}{w} \Rightarrow y' = -\dfrac{1}{w^2} \cdot w' \Rightarrow$

$$-\frac{1}{w^2} \cdot w' - \frac{1}{w} = x \cdot \frac{1}{w^2} \Rightarrow w' + w = -x \Rightarrow$$

$$\Rightarrow w = e^{-\int dx} \cdot \left[\int -x \, e^{\int dx} dx + k \right] = e^{-x} \cdot \left(-\left(\int x e^x dx \right) + k \right) =$$

$$= e^{-x} \cdot \left(-x e^x + e^x + k \right) = -x + 1 + k \cdot e^{-x} = w \Rightarrow$$

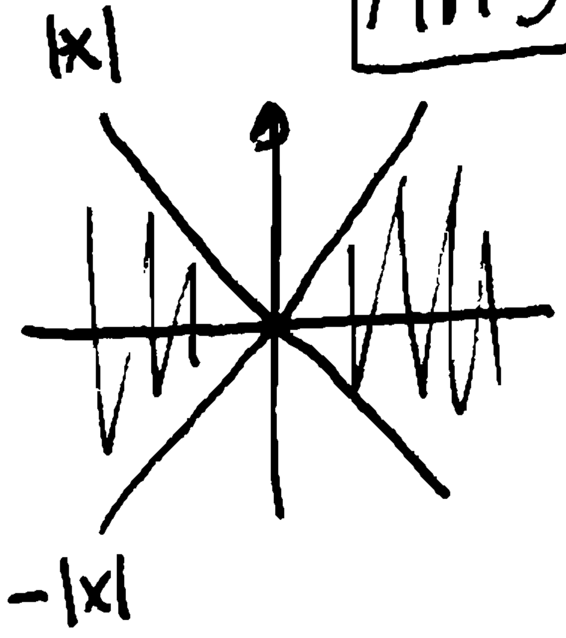
$$\Rightarrow y = \frac{1}{w} = \frac{1}{k e^{-x} - x + 1}.$$

$$y(0) = 1 \Rightarrow 1 = \frac{1}{k - 0 + 1} \Rightarrow \frac{1}{k+1} = 1 \Rightarrow k = 0.$$

Soluzione unica del problema di Cauchy: $y = \dfrac{1}{1 - x}$.

AM5

II M4) $f(x;y) = \log(x^2-y^2)$.



$$\text{e.E.}: x^2-y^2>0 \Rightarrow y^2<x^2 \Rightarrow -\sqrt{x^2}<y<\sqrt{x^2} \Rightarrow -|x|<y<|x|.$$

$$\nabla f(x;y) = \left(\frac{2x}{x^2-y^2}\,;\,\frac{-2y}{x^2-y^2}\right).$$

$$\|\nabla f(x;y)\|^2 = \frac{4x^2}{(x^2-y^2)^2}+\frac{4y^2}{(x^2-y^2)^2} = \frac{4(x^2+y^2)}{(x^2-y^2)^2}.$$

$$H(f(x;y)) = \begin{Vmatrix} \frac{2(x^2-y^2)-4x^2}{(x^2-y^2)^2} & \frac{4xy}{(x^2-y^2)^2} \\ \frac{4xy}{(x^2-y^2)^2} & \frac{-2(x^2-y^2)-4y^2}{(x^2-y^2)^2} \end{Vmatrix} = \begin{Vmatrix} -\frac{2(x^2+y^2)}{(x^2-y^2)^2} & \frac{4xy}{(x^2-y^2)^2} \\ \frac{4xy}{(x^2-y^2)^2} & -\frac{2(x^2+y^2)}{(x^2-y^2)^2} \end{Vmatrix}.$$

$$\|\nabla f\|^2 = |H| \Rightarrow$$

$$\Rightarrow \frac{\cancel{4}(x^2+y^2)}{(x^2-y^2)^2} = \frac{\cancel{4}(x^2+y^2)^2}{(x^2-y^2)^4} - \frac{\overset{4}{\cancel{16}}x^2y^2}{(x^2-y^2)^4} \Rightarrow \frac{x^2+y^2}{\cancel{(x^2-y^2)^2}} = \frac{x^4+y^4+2x^2y^2-4x^2y^2}{(x^2-y^2)^{\cancel{4}\,2}} \Rightarrow$$

$$\Rightarrow x^2+y^2 = \frac{x^4+y^4-2x^2y^2}{(x^2-y^2)^2} = \frac{(x^2-y^2)^2}{(x^2-y^2)^2} = 1 \Rightarrow$$

$$\Rightarrow x^2+y^2=1 \text{ con } -|x|<y<|x|$$

Soluzione: