Compito di Analisi Matematica del 22/3/14 $\boxed{AM1}$

I M1) $\frac{1}{2}+\frac{1}{2}i = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \frac{1}{\sqrt{2}}\cdot\left(\cos\frac{\pi}{4}+i\,\text{sen}\,\frac{\pi}{4}\right).$

$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10} = \frac{1}{(\sqrt{2})^{10}}\cdot\left(\cos\left(10\cdot\frac{\pi}{4}\right)+i\,\text{sen}\left(10\cdot\frac{\pi}{4}\right)\right) = \frac{1}{2^5}\cdot\left(\cos\frac{5}{2}\pi+i\,\text{sen}\,\frac{5}{2}\pi\right) =$

$\left(\frac{1}{2}+\frac{1}{2}i\right)^{10} = 2^{-5}\cdot\left(\cos\frac{\pi}{2}+i\,\text{sen}\,\frac{\pi}{2}\right) = 2^{-5}\cdot(0+1\cdot i) = 2^{-5}\cdot i.$

$\sqrt[3]{\left(\frac{1}{2}+\frac{1}{2}i\right)^{10}} = \sqrt[3]{2^{-5}}\cdot\left(\cos\left(\frac{\pi}{6}+K\cdot\frac{2\pi}{3}\right)+i\,\text{sen}\left(\frac{\pi}{6}+K\cdot\frac{2\pi}{3}\right)\right);\ 0\leq K\leq 2.$

per $K=0$: $\frac{1}{\sqrt[3]{2^5}}\cdot\left(\cos\frac{\pi}{6}+i\,\text{sen}\,\frac{\pi}{6}\right) = \frac{1}{2\sqrt[3]{4}}\cdot\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right);$

per $K=1$: $\frac{1}{\sqrt[3]{2^5}}\cdot\left(\cos\frac{5}{6}\pi+i\,\text{sen}\,\frac{5}{6}\pi\right) = \frac{1}{2\sqrt[3]{4}}\cdot\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right);$

per $K=2$: $\frac{1}{\sqrt[3]{2^5}}\cdot\left(\cos\frac{9}{6}\pi+i\,\text{sen}\,\frac{9}{6}\pi\right) = \frac{1}{2\sqrt[3]{4}}\cdot\left(\cos\frac{3}{2}\pi+i\,\text{sen}\,\frac{3}{2}\pi\right) = -\frac{1}{2\sqrt[3]{4}}\cdot i.$

I M2) $f(x;y)=\begin{cases}\frac{x^2|y|^\alpha}{(x^2+y^2)^\alpha} & :(x;y)\neq(0;0)\\ 0 & :(x;y)=(0;0)\end{cases}\ .\ \alpha>0.$

$\lim_{(x;y)\to(0;0)}\frac{x^2|y|^\alpha}{(x^2+y^2)^\alpha} \Rightarrow \lim_{\rho\to 0}\frac{\rho^{2+\alpha}\cdot\cos^2\vartheta\cdot|\text{sen}\,\vartheta|^\alpha}{\rho^{2\alpha}} = 0$ se $2+\alpha>2\alpha \Rightarrow$

$\Rightarrow \alpha<2$. Dato che $|\cos^2\vartheta\cdot|\text{sen}\,\vartheta|^\alpha|<1$, la funzione è continua se $0\leq\alpha<2$.

$\frac{\partial f}{\partial x}(0;0) = \lim_{h\to 0}\left(\frac{h^2\cdot 0}{(h^2+0)^\alpha}-0\right)\cdot\frac{1}{h} = \lim_{h\to 0}\frac{0}{h} = 0;$

$\frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0}\left(\frac{0\cdot|h|^\alpha}{(0+h^2)^\alpha}-0\right)\cdot\frac{1}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$ quindi

$\lim_{(x;y)\to(0;0)}\frac{\frac{x^2|y|^\alpha}{(x^2+y^2)^\alpha}-0-(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}} \Rightarrow \lim_{\rho\to 0}\frac{\rho^{2+\alpha}\cos^2\vartheta\cdot|\text{sen}\,\vartheta|^\alpha}{\rho^{2\alpha}\cdot\rho} =$

$= \lim_{\rho\to 0}\rho^{2+\alpha-2\alpha-1}\cdot\cos^2\vartheta\cdot|\text{sen}\,\vartheta|^\alpha = \lim_{\rho\to 0}\rho^{1-\alpha}\cdot\cos^2\vartheta\cdot|\text{sen}\,\vartheta|^\alpha = 0$

se $1-\alpha>0 \Rightarrow 0\leq\alpha<1$. f è differenziabile in $(0;0)$ se $0\leq\alpha<1$.

I M3) $\begin{cases} f(x;y;z;w) = x^3y-y^2z^2+zyw^2=1\\ g(x;y;z;w) = e^xy-ze^w+x^2zw=1\end{cases};\ \begin{cases} f(1;1;1;1)=1\\ g(1;1;1;1)=1\end{cases}.$

$$\frac{\partial(f;g)}{\partial(x,y,z,w)} = \left\|\begin{matrix} 3x^2y & x^3-2yz^2+zw^2 & -2y^2z+yw^2 & 2zyw\\ e^xy+2xzw & e^x & -e^w+x^2w & -ze^w+x^2z\end{matrix}\right\|$$

$\boxed{AM2}$

$$\frac{\partial(f;g)}{\partial(x,y,z,w)}(1;1;1;1) = \left\| \begin{matrix} 3 & 0 & -1 & 2 \\ e+2 & e & 1-e & 1-e \end{matrix} \right\|.$$

Dato che $\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix} = -1+e-2+2e = 3e-3 \neq 0$ è possibile definire una funzione implicita $(x;y) \to (z(x;y);\, w(x;y))$.

$$\frac{\partial z}{\partial x} = -\frac{\begin{vmatrix} 3 & 2 \\ e+2 & 1-e \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{3-3e-2e-4}{3e-3} = \frac{5e+1}{3e-3};\quad \frac{\partial z}{\partial y} = -\frac{\begin{vmatrix} 0 & 2 \\ e & 1-e \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{0-2e}{3e-3} = \frac{2e}{3e-3};$$

$$\frac{\partial w}{\partial x} = -\frac{\begin{vmatrix} -1 & 3 \\ 1-e & e+2 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{-e-2-3+3e}{3e-3} = \frac{5-2e}{3e-3};\quad \frac{\partial w}{\partial y} = -\frac{\begin{vmatrix} -1 & 0 \\ 1-e & e \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{-e}{3e-3} = \frac{e}{3e-3}.$$

IM4) $f_n(x) = x \cdot \log^{2n} x$. C.E.: $x > 0$

$$\lim_{n\to+\infty} f_n(x) = \begin{cases} \text{se } |\log x| < 1 \Rightarrow \frac{1}{e} < x < e : \lim = 0 \\ \text{se } |\log x| > 1 \Rightarrow 0 < x < \frac{1}{e} \cup e < x : \lim = +\infty \\ f_n\left(\frac{1}{e}\right) = \frac{1}{e}\cdot(-1)^{2n} = \frac{1}{e} \Rightarrow \lim = \frac{1}{e} \\ f_n(e) = e \Rightarrow \lim = e \end{cases}$$

$C = \left[\frac{1}{e}; e\right]$; $f(x) = \begin{cases} \frac{1}{e} : x = \frac{1}{e} \\ 0 : \frac{1}{e} < x < e \\ e : x = e \end{cases}$. Studiamo $f_n(x)$. $f_n(1) = 0$.

$$f_n'(x) = 1\cdot \log^{2n} x + \cancel{x}\cdot 2n \log^{2n-1} x \cdot \frac{1}{\cancel{x}} = \log^{2n-1} x \cdot (\log x + 2n) \geq 0$$

Dato che $2n-1$ è dispari $\forall n \geq 1$ avremo:

$\log^{2n-1} x \geq 0$ per $\log x \geq 0 \Rightarrow x \geq 1$

$\log x + 2n \geq 0$ per $\log x \geq -2n \Rightarrow x \geq e^{-2n} = \frac{1}{e^{2n}}$

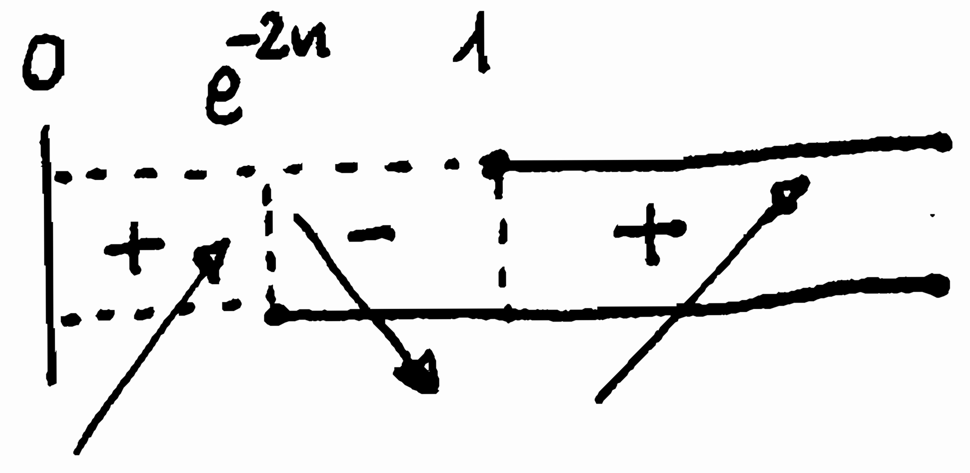
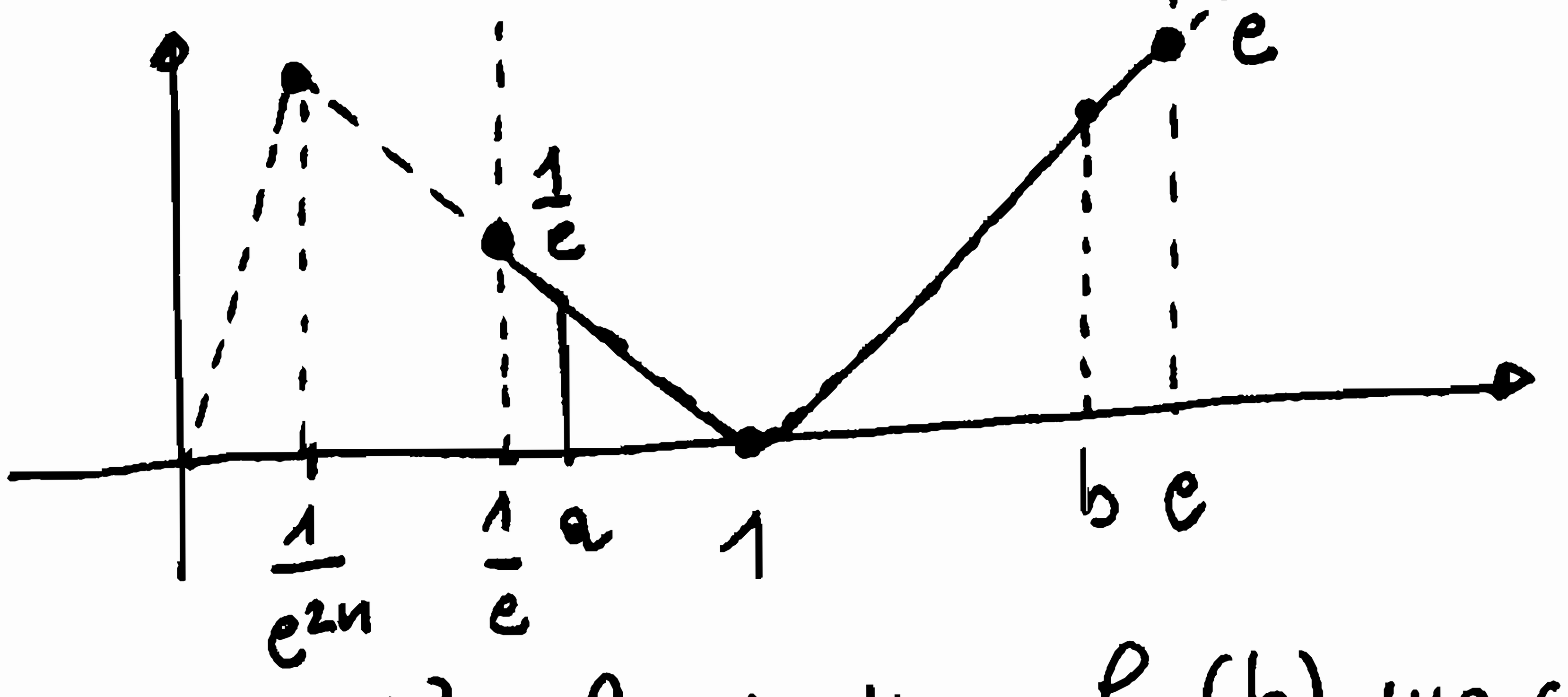


Grafico:



Preso $[a;b]$ con $a > \frac{1}{e}$ e $b < e$

$\sup_{x\in[a;b]} \{|f_n(x) - 0|\} = f_n(a)$ oppure $f_n(b)$ ma comunque risulta

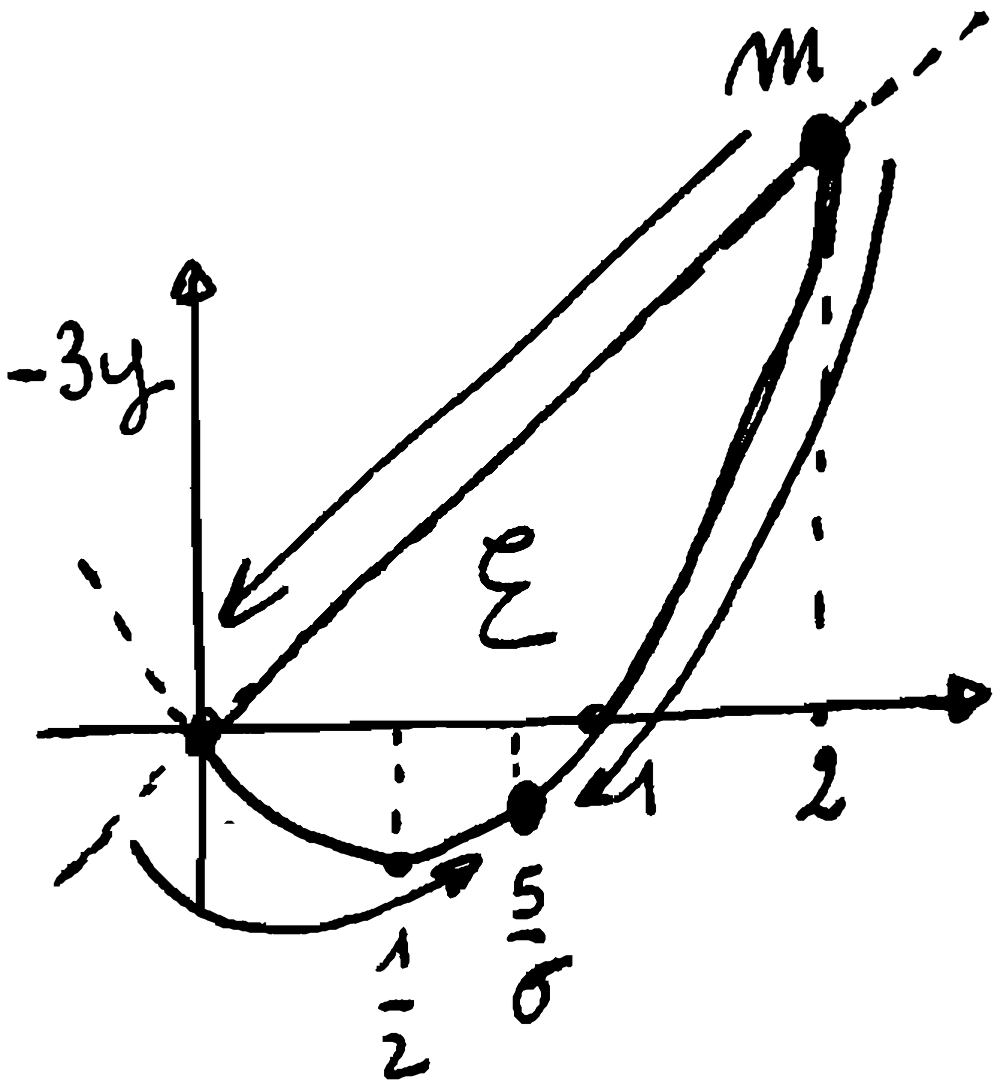
$\lim_{n\to+\infty} f_n(a) = \lim_{n\to+\infty} f_n(b) = 0$ e quindi la convergenza è uniforme in ogni $[a;b] \subset \left]\frac{1}{e}; e\right[$.

Risulta poi $\sum_{n=0}^{+\infty} x \cdot \log^{2n} x = x \cdot \sum_{n=0}^{+\infty} (\log^2 x)^n$ ovvero una serie geometrica di ragione $\log^2 x$. Dovrà quindi essere: **AM3**

$\log^2 x < 1 \Rightarrow -1 < \log x < 1 \Rightarrow \frac{1}{e} < x < e$ per avere una serie convergente.

Avremo poi $S(x) = x \cdot \frac{1}{1-\log^2 x} = \frac{x}{1-\log^2 x}$.

II M1) $\begin{cases} \text{Max/min } f(x;y) = 2x-3y \\ \text{s.v.} \begin{cases} y \geq x^2 - x \\ y \leq x \end{cases} \end{cases} \Rightarrow \begin{cases} \text{Max/min } f(x;y) = 2x-3y \\ \text{s.v.} \begin{cases} x^2-x-y \leq 0 \\ y-x \leq 0 \end{cases} \end{cases}$



$f(x;y)$ continua e differenziabile.

Σ insieme limitato e chiuso.

Vincoli ovunque qualificati.

$\Lambda = 2x - 3y - \lambda_1 (x^2 - x - y) - \lambda_2 (y - x)$

Caso $\lambda_1 = \lambda_2 = 0$

$\begin{cases} \Lambda'_x = 2 \neq 0 \\ \Lambda'_y = -3 \neq 0 \end{cases}$ non ci sono soluzioni.

Caso $\lambda_1 \neq 0; \lambda_2 = 0$

$\begin{cases} \Lambda'_x = 2 - 2\lambda_1 x + \lambda_1 = 0 \\ \Lambda'_y = -3 + \lambda_1 = 0 \\ y = x^2 - x \\ y \leq x \end{cases} \Rightarrow \begin{cases} \lambda_1 = 3 \\ 2 - 6x + 3 = 0 \Rightarrow x = \frac{5}{6} \\ y = \frac{25}{36} - \frac{5}{6} = -\frac{5}{36} \\ -\frac{5}{36} \leq \frac{5}{6} : \text{vera} \end{cases}$ Max ?

Caso $\lambda_1 = 0; \lambda_2 \neq 0$

$\begin{cases} \Lambda'_x = 2 + \lambda_2 = 0 \\ \Lambda'_y = -3 - \lambda_2 = 0 \\ y = x \\ y \geq x^2 - x \end{cases} \Rightarrow \begin{cases} \lambda_2 = -2 \\ \lambda_2 = -3 \end{cases}$: Impossibile

Caso $\lambda_1 \neq 0; \lambda_2 \neq 0$

$\begin{cases} \Lambda'_x = 2 - 2\lambda_1 x + \lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = -3 + \lambda_1 - \lambda_2 = 0 \\ \begin{cases} y = x \\ y = x^2 - x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{e} \begin{cases} x = 2 \\ y = 2 \end{cases} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 2 + \lambda_1 + \lambda_2 = 0 \\ -3 + \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 2\lambda_1 = 1 \Rightarrow \lambda_1 = \frac{1}{2} > 0 \\ \lambda_2 = -3 + \lambda_1 = -\frac{5}{2} < 0 \end{cases}$: **Nulla**

$$\begin{cases} x=2 \\ y=2 \\ 2-3\lambda_1+\lambda_2=0 \\ -3+\lambda_1-\lambda_2=0 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=2 \\ 2\lambda_1=-1 \Rightarrow \lambda_1=-\frac{1}{2}<0 \\ \lambda_2=\lambda_1-3=-\frac{7}{2}<0 \end{cases}$$

Min ?

AM4

Per il Teorema di Weierstrass $\left(\frac{5}{6};-\frac{5}{36}\right)$ è il punto di Massimo Assoluto mentre $(2;2)$ è il punto di minimo assoluto.

II M2) $\begin{cases} x'=2x+3y+e^t \\ y'=-x-2y+1 \end{cases} \Rightarrow \begin{cases} x'-2x-3y=e^t \\ x+y'+2y=1 \end{cases} \Rightarrow \left\|\begin{matrix} D-2 & -3 \\ 1 & D+2 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \end{matrix}\right\| = \left\|\begin{matrix} e^t \\ 1 \end{matrix}\right\|.$

$\Rightarrow \begin{vmatrix} D-2 & -3 \\ 1 & D+2 \end{vmatrix}(x) = \begin{vmatrix} e^t & -3 \\ 1 & D+2 \end{vmatrix} \Rightarrow (D^2-4+3)(x)=(D+2)(e^t)+3 \Rightarrow (D^2-1)(x)=3e^t+3 \Rightarrow$

$\Rightarrow x''-x=3e^t+3.$ $\lambda^2-1=0 \Rightarrow \lambda=\pm 1$ per cui l'equazione omogenea ha per soluzione generale $x(t)=c_1e^t+c_2e^{-t}$.

Preso $x_0(t)=a\cdot e^t+bte^t+c$ $((D-1)$ annichila $e^t\ldots)$ avremo

$x_0'=ae^t+be^t+bte^t=(a+b)e^t+bte^t;$

$x_0''=(a+b)e^t+be^t+bte^t=(a+2b)e^t+bte^t;$ sostituendo:

$(a+2b)e^t+\cancel{bte^t}-ae^t-\cancel{bte^t}-c=3e^t+3 \Rightarrow$

$\Rightarrow \begin{cases} a+2b-a=3 \\ -c=+3 \end{cases} \Rightarrow \begin{cases} 2b=3 \\ c=-3 \end{cases} \Rightarrow \begin{cases} b=\frac{3}{2} \\ c=-3 \end{cases}$ per avere quindi:

$x(t)=c_1e^t+c_2e^{-t}+\frac{3}{2}te^t-3.$ Da $y=\frac{1}{3}(x'-2x-e^t)$ si ha poi:

$y(t)=\frac{1}{3}\left(c_1e^t-c_2e^{-t}+\frac{3}{2}e^t+\frac{3}{2}te^t-2c_1e^t-2c_2e^{-t}-3te^t+6-e^t\right)=$

$y(t)=-\frac{1}{3}c_1e^t+\frac{1}{6}e^t-c_2e^{-t}-\frac{1}{2}te^t+2=\frac{1}{3}\left(\frac{1}{2}-c_1\right)e^t-c_2e^{-t}-\frac{1}{2}te^t+2.$

II M3) $\begin{cases} y'=(1+y^2)\cdot x \\ y(0)=1 \end{cases}$. Equazione a variabili separabili.

$\frac{1}{1+y^2}\cdot y'=x \Rightarrow \int\frac{1}{1+y^2}dy=\int x\,dx+k \Rightarrow \operatorname{arctg} y=\frac{x^2}{2}+k \Rightarrow$

$\Rightarrow y=\operatorname{tg}\left(\frac{x^2}{2}+k\right).$

$y(0)=1 \Rightarrow 1=\operatorname{tg}k \Rightarrow k=\operatorname{arctg}1=\frac{\pi}{4}.$

Soluzione del problema: $y(x)=\operatorname{tg}\left(\frac{x^2}{2}+\frac{\pi}{4}\right).$

II M4) $f(x;y)=\frac{1}{2}e^{2-x^2-y^2}$. Funzione differenziabile due volte. **AM5**

$\nabla f(x;y)=\left(\frac{1}{2}\cdot(-2x)e^{2-x^2-y^2};\frac{1}{2}\cdot(-2y)\cdot e^{2-x^2-y^2}\right)=\left(-xe^{2-x^2-y^2};-ye^{2-x^2-y^2}\right)$.

$$\begin{cases}\mathcal{D}_v f(P)=\nabla f(P)\cdot(1;0)=-x\cdot e^{2-x^2-y^2}=1\\ \mathcal{D}_w f(P)=\nabla f(P)\cdot(0;1)=-y\,e^{2-x^2-y^2}=-1\end{cases}\Rightarrow\begin{cases}-x\,e^{2-x^2-y^2}=1\\-y\,e^{2-x^2-y^2}=-1\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}y=-x\\-x\,e^{2-2x^2}=1\end{cases}\Rightarrow\begin{cases}y=1\\x=-1\end{cases}$$. Un punto P può essere $(-1;1)$.

In realtà ci sono due soluzioni, ma ce ne basta una.

$$H(f)=\begin{Vmatrix}-e^{2-x^2-y^2}+2x^2e^{2-x^2-y^2} & 2xye^{2-x^2-y^2}\\ 2xy\,e^{2-x^2-y^2} & -e^{2-x^2-y^2}+2y^2e^{2-x^2-y^2}\end{Vmatrix}$$

$$H(f)=\begin{Vmatrix}(2x^2-1)\,e^{2-x^2-y^2} & 2xy\,e^{2-x^2-y^2}\\ 2xy\,e^{2-x^2-y^2} & (2y^2-1)\,e^{2-x^2-y^2}\end{Vmatrix}$$

$$H(-1;1)=\begin{Vmatrix}1 & -2\\-2 & 1\end{Vmatrix}.$$

$$\mathcal{D}^2_{v,w}(-1;1)=\begin{Vmatrix}1;0\end{Vmatrix}\cdot\begin{Vmatrix}1 & -2\\-2 & 1\end{Vmatrix}\cdot\begin{Vmatrix}0\\1\end{Vmatrix}=\begin{Vmatrix}1;0\end{Vmatrix}\cdot\begin{Vmatrix}-2\\1\end{Vmatrix}=-2.$$