Compito di Analisi Matematica del 4/7/2014 $\boxed{AM1}$

IM1) $\sqrt{(1-i)^3}$. $1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\cdot\left(\cos\frac{7}{4}\pi + i \operatorname{sen}\frac{7}{4}\pi\right)$.

$(1-i)^3 = \sqrt{8}\cdot\left(\cos\frac{21}{4}\pi + i\operatorname{sen}\frac{21}{4}\pi\right) = 2\sqrt{2}\left(\cos\frac{5}{4}\pi + i\operatorname{sen}\frac{5}{4}\pi\right)$.

$\sqrt{(1-i)^3} = \sqrt[4]{8}\cdot\left(\cos\left(\frac{5}{8}\pi + k\cdot\frac{2\pi}{2}\right) + i\operatorname{sen}\left(\frac{5}{8}\pi + k\cdot\frac{2\pi}{2}\right)\right)$; $0 \leq k \leq 1$.

Per $k=0$: $\sqrt[4]{8}\cdot\left(\cos\frac{5}{8}\pi + i\operatorname{sen}\frac{5}{8}\pi\right)$;

Per $k=1$: $\sqrt[4]{8}\cdot\left(\cos\frac{13}{8}\pi + i\operatorname{sen}\frac{13}{8}\pi\right)$.

IM2) $\begin{cases} f(x;y;z) = (x+y+z)\cdot e^{x+y+z} = 0 \\ g(x;y;z) = x\cdot e^{yz} + y\cdot e^{xz} + z\cdot e^{xy} = 0 \end{cases} \Rightarrow \begin{cases} f(1;0;-1) = 0\cdot 1 = 0 \\ g(1;0;-1) = 1+0-1 = 0 \end{cases}$.

$$\frac{\partial(f;g)}{\partial(x;y;z)} = \left\|\begin{matrix} (x+y+z+1)\cdot e^{x+y+z} & (x+y+z+1)e^{x+y+z} & (x+y+z+1)e^{x+y+z} \\ e^{yz}+yze^{xz}+yze^{xy} & xze^{yz}+e^{xz}+xze^{xy} & xye^{yz}+xye^{xz}+e^{xy} \end{matrix}\right\|$$

$$\frac{\partial(f;g)}{\partial(x;y;z)}(1;0;-1) = \left\|\begin{matrix} 1 & 1 & 1 \\ 1 & \frac{1}{e}-2 & 1 \end{matrix}\right\|.$$ Dato che $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$ non è possibile definire una funzione implicita $y \to (x(y); z(y))$.

Dato che $\begin{vmatrix} 1 & 1 \\ \frac{1}{e}-2 & 1 \end{vmatrix} = 1 - \frac{1}{e} + 2 = 3 - \frac{1}{e} \neq 0$ si può definire $F: x \to (y(x); z(x))$.

$$\frac{dy}{dx} = -\frac{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \frac{1}{e}-2 & 1 \end{vmatrix}} = 0 \ ; \quad \frac{dz}{dx} = -\frac{\begin{vmatrix} 1 & 1 \\ \frac{1}{e}-2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \frac{1}{e}-2 & 1 \end{vmatrix}} = -1.$$

Equazione retta tangente in $x=1$: $x \to (0;-1) + x\cdot(0;-1) = (0;-1-x)$.

IM3) $f(x;y) = x\cdot|\operatorname{sen} y|$; $f(0;0) = 0$ e la funzione è continua in $(0;0)$ in quanto prodotto di funzioni continue.

$$\frac{\partial f}{\partial x}(0;0) = \lim_{h\to 0}\frac{(0+h)\cdot|\operatorname{sen} 0| - 0}{h} = \lim_{h\to 0}\frac{0}{h} = 0;$$

$$\frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0}\frac{0\cdot|\operatorname{sen}(0+h)| - 0}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Occorre quindi verificare se:

$$\lim_{(x;y)\to(0;0)} \frac{x\cdot|\operatorname{sen} y| - 0 - (0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}$$ risulti uguale a 0.

Passando a coordinate polari:

$$\Rightarrow \lim_{\rho\to 0} \frac{\rho\cos\vartheta\cdot|\operatorname{sen}(\rho\operatorname{sen}\vartheta)|-0-0}{\rho} = \lim_{\rho\to 0} \cos\vartheta\cdot|\operatorname{sen}(\rho\operatorname{sen}\vartheta)| = 0 \quad \boxed{AM2}$$

in quanto $\rho\to 0 \Rightarrow \rho\cdot\operatorname{sen}\vartheta\to 0 \Rightarrow \operatorname{sen}(\rho\cdot\operatorname{sen}\vartheta)\to 0$.

Dato che $|\operatorname{sen}\alpha|\leq|\alpha|\ \forall\alpha\in\mathbb{R}$, segue che:

$$|\cos\vartheta\cdot|\operatorname{sen}(\rho\cdot\operatorname{sen}\vartheta)|-0|\leq|\cos\vartheta|\cdot|\operatorname{sen}(\rho\operatorname{sen}\vartheta)|\leq|\cos\vartheta|\cdot|\rho\operatorname{sen}\vartheta|\leq$$

$\leq \rho\cdot 1\cdot 1 < \varepsilon$ se $\rho<\varepsilon$ e quindi la convergenza del limite è uniforme. Quindi la funzione è differenziabile in $(0;0)$.

IM4) $\displaystyle\sum_{n=1}^{+\infty}\frac{3^{n-2}}{n^3\cdot 2^{n+2}}\cdot x^n$. Centro nel punto $x_0=0$.

$$\rho=\lim_{n\to+\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to+\infty}\frac{3^{n-2}}{n^3\cdot 2^{n+2}}\cdot\frac{(n+1)^3\cdot 2^{n+3}}{3^{n-1}} = \lim_{n\to+\infty}\left(\frac{n+1}{n}\right)^3\cdot\frac{2}{3}=\frac{2}{3}.$$

Se $x=-\frac{2}{3}$: $\displaystyle\sum_{n=1}^{+\infty}\frac{3^{n-2}}{n^3\cdot 2^{n+2}}\cdot(-1)^n\cdot\frac{2^n}{3^n}=\sum_{n=1}^{+\infty}(-1)^n\cdot\frac{1}{n^3}\cdot\frac{1}{4\cdot 9}$: Convergente (Leibnitz)

Se $x=\frac{2}{3}$: $\displaystyle\sum_{n=1}^{+\infty}\frac{3^{n-2}}{n^3\cdot 2^{n+2}}\cdot\frac{2^n}{3^n}=\sum_{n=1}^{+\infty}\frac{1}{4\cdot 9}\cdot\frac{1}{n^3}=\frac{1}{36}\cdot\sum_{n=1}^{+\infty}\frac{1}{n^3}$: Convergente $(\alpha=3>1)$.

Quindi $\mathcal{C}=\left[-\frac{2}{3};\frac{2}{3}\right]$.

II M1) $\begin{cases}\text{Max } f(x;y;z)=x+y+z\\ \text{S.V.}: x^2+2y^2+3z^2=11\end{cases}$.

$\Lambda = x+y+z-\lambda(x^2+2y^2+3z^2-11)$.

$$\begin{cases}\Lambda'_x = 1-2\lambda x=0\\ \Lambda'_y = 1-4\lambda y=0\\ \Lambda'_z = 1-6\lambda z=0\\ x^2+2y^2+3z^2=11\end{cases}\Rightarrow\begin{cases}x=\frac{1}{2\lambda}\\ y=\frac{1}{4\lambda}\\ z=\frac{1}{6\lambda}\\ \frac{1}{\lambda^2}\left(\frac{1}{4}+\frac{2}{16}+\frac{3}{36}\right)=11\end{cases}\Rightarrow\begin{cases}x=\sqrt{6}\\ y=\frac{\sqrt{6}}{2}\\ z=\frac{\sqrt{6}}{3}\\ \lambda^2=\frac{11}{24}\cdot\frac{1}{11}=\frac{1}{24}\Rightarrow\lambda=+\frac{1}{2\sqrt{6}}\end{cases}$$

La soluzione è data dal punto $P=\left(\sqrt{6};\frac{\sqrt{6}}{2};\frac{\sqrt{6}}{3}\right)$ con $\lambda=\frac{1}{2\sqrt{6}}$.

Verifichiamo la natura del punto P.

$$\overline{H}(x;y;z;\lambda)=\begin{Vmatrix}0 & 2x & 4y & 6z\\ 2x & -2\lambda & 0 & 0\\ 4y & 0 & -4\lambda & 0\\ 6z & 0 & 0 & -6\lambda\end{Vmatrix};\ \overline{H}(P)=\begin{Vmatrix}0 & 2\sqrt{6} & 2\sqrt{6} & 2\sqrt{6}\\ 2\sqrt{6} & -\frac{1}{\sqrt{6}} & 0 & 0\\ 2\sqrt{6} & 0 & -\frac{2}{\sqrt{6}} & 0\\ 2\sqrt{6} & 0 & 0 & -\frac{3}{\sqrt{6}}\end{Vmatrix}$$

AM3

$$|\overline{H}_3(P)| = \begin{vmatrix} 0 & 2\sqrt6 & 2\sqrt6 \\ 2\sqrt6 & -\frac{1}{\sqrt6} & 0 \\ 2\sqrt6 & 0 & -\frac{2}{\sqrt6} \end{vmatrix} = \begin{vmatrix} 0 & 2\sqrt6 & 2\sqrt6 \\ 0 & -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} \\ 2\sqrt6 & 0 & -\frac{2}{\sqrt6} \end{vmatrix} = 2\sqrt6 \cdot \begin{vmatrix} 2\sqrt6 & 2\sqrt6 \\ -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} \end{vmatrix} = 2\sqrt6\cdot(4+2) > 0;$$

$$|\overline{H}_4(P)| = \begin{vmatrix} 0 & 2\sqrt6 & 2\sqrt6 & 2\sqrt6 \\ 2\sqrt6 & -\frac{1}{\sqrt6} & 0 & 0 \\ 2\sqrt6 & 0 & -\frac{2}{\sqrt6} & 0 \\ 2\sqrt6 & 0 & 0 & -\frac{3}{\sqrt6} \end{vmatrix} = \begin{vmatrix} 0 & 2\sqrt6 & 2\sqrt6 & 2\sqrt6 \\ 0 & -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} & 0 \\ 2\sqrt6 & 0 & -\frac{2}{\sqrt6} & 0 \\ 0 & \frac{1}{\sqrt6} & 0 & -\frac{3}{\sqrt6} \end{vmatrix} = 2\sqrt6 \cdot \begin{vmatrix} 2\sqrt6 & 2\sqrt6 & 2\sqrt6 \\ -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} & 0 \\ \frac{1}{\sqrt6} & 0 & -\frac{3}{\sqrt6} \end{vmatrix} =$$

$$= 2\sqrt6 \cdot \begin{vmatrix} 2\sqrt6 & 0 & 0 \\ -\frac{1}{\sqrt6} & \frac{3}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & -\frac{4}{\sqrt6} \end{vmatrix} = 2\sqrt6 \cdot 2\sqrt6 \cdot \begin{vmatrix} \frac{3}{\sqrt6} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt6} & -\frac{4}{\sqrt6} \end{vmatrix} = 24\cdot\left(-2+\frac{1}{6}\right) = -44 < 0.$$

Da $(\overline{H}_3 > 0;\ \overline{H}_4 < 0)$ segue che P è un punto di Massimo.

II M2) $\begin{cases} x' = x + 2y + e^t \\ y' = 3x + 2y - e^t \end{cases} \Rightarrow \begin{cases} x' - x - 2y = e^t \\ -3x + y' - 2y = -e^t \end{cases} \Rightarrow \begin{cases} (D-1)x - 2y = e^t \\ -3x + (D-2)y = -e^t \end{cases} \Rightarrow$

$$\Rightarrow \begin{vmatrix} D-1 & -2 \\ -3 & D-2 \end{vmatrix}(x) = \begin{vmatrix} e^t & -2 \\ -e^t & D-2 \end{vmatrix} \Rightarrow (D^2-3D-4)(x) = e^t - 2e^t - 2e^t = -3e^t.$$

$(D^2-3D-4) = (D-4)(D+1) \Rightarrow x(t) = c_1 e^{4t} + c_2 e^{-t}$: Soluzione eq. Omogenea.

Posto $x_0(t) = a\cdot e^t \Rightarrow x_0' = x_0'' = a e^t \Rightarrow a e^t - 3a e^t - 4a e^t = -3e^t \Rightarrow$

$\Rightarrow -6a e^t = -3e^t \Rightarrow a = \frac{1}{2}.$

Soluzione generale per $x(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{1}{2} e^t.$

Da $y = \frac{1}{2}(x' - x - e^t) = \frac{1}{2}\left(4c_1 e^{4t} - c_2 e^{-t} + \frac{1}{2} e^t - c_1 e^{4t} - c_2 e^{-t} - \frac{1}{2} e^t - e^t\right) =$

$= y = \frac{1}{2}\left(3c_1 e^{4t} - 2c_2 e^{-t} - e^t\right) \Rightarrow y(t) = \frac{3}{2} c_1 e^{4t} - c_2 e^{-t} - \frac{1}{2} e^t.$

II M3) $\begin{cases} e^x \cdot y' = e^y \cdot x \\ y(0) = 1 \end{cases} \Rightarrow e^{-y}\cdot y' = x\cdot e^{-x} \Rightarrow \int e^{-y}\,dy = \int x\cdot e^{-x}\,dx + K \Rightarrow$

$\Rightarrow -e^{-y} = -x e^{-x} - \int 1\cdot(-e^{-x})\,dx + K = -x e^{-x} + \int e^{-x}\,dx + K \Rightarrow$

$\Rightarrow -e^{-y} = -x e^{-x} - e^{-x} + K \Rightarrow e^{-y} = x e^{-x} + e^{-x} + m \quad (m = -K)$

AM4

$\Rightarrow -y = \log\left(xe^{-x}+e^{-x}+m\right) \Rightarrow$

$\Rightarrow y = -\log\left(xe^{-x}+e^{-x}+m\right) = \log\dfrac{1}{xe^{-x}+e^{-x}+m}$.

Imponendo $y(0)=1$ si ha: $1 = \log\dfrac{1}{0+1+m} \Rightarrow$

$\Rightarrow 1 = \log\dfrac{1}{1+m} \Rightarrow \dfrac{1}{1+m} = e \Rightarrow 1+m = \dfrac{1}{e} \Rightarrow m = \dfrac{1}{e}-1.$

Soluzione del problema: $y(x) = \log\dfrac{1}{xe^{-x}+e^{-x}+\frac{1}{e}-1}$.

II M4) $f(x;y) = x^3+y^3$: funzione sempre differenziabile.

$u = \left(\frac{1}{\sqrt2}; -\frac{1}{\sqrt2}\right)$; $v = \left(\frac{1}{\sqrt2}; \frac{1}{\sqrt2}\right)$.

$\mathcal{D}_u f(x;y) = \nabla f(x;y)\cdot u = (3x^2; 3y^2)\cdot\left(\frac{1}{\sqrt2}; -\frac{1}{\sqrt2}\right) = 0$;

$\mathcal{D}_v f(x;y) = \nabla f(x;y)\cdot v = (3x^2; 3y^2)\cdot\left(\frac{1}{\sqrt2}; \frac{1}{\sqrt2}\right) = 3\sqrt2$; $\Rightarrow$

$$\begin{cases}\frac{3}{\sqrt2}(x^2-y^2)=0\\ \frac{3}{\sqrt2}(x^2+y^2)=3\sqrt2\end{cases} \Rightarrow \begin{cases}x^2-y^2=0\\ x^2+y^2=2\end{cases} \Rightarrow \begin{cases}y=\pm x\\ 2x^2=2\end{cases} \Rightarrow \begin{cases}y=\pm x\\ x^2=1\end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases}y=\pm x\\ x=\pm 1\end{cases} \Rightarrow$$ 4 Soluzioni: $(1;1)$; $(1;-1)$; $(-1;1)$; $(-1;-1)$.