Compito di Analisi Matematica del 15/11/2014

AM1

IM1) $x^3-3x^2+3x-2=0$. Se $x=2$: $8-12+6-2=0$.

$$\begin{array}{c|ccc|c} & 1 & -3 & 3 & -2 \\ 2 & & 2 & -2 & 2 \\ \hline & 1 & -1 & 1 & 0 \end{array} \Rightarrow (x-2)(x^2-x+1)=0 \Rightarrow x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{-3}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$$

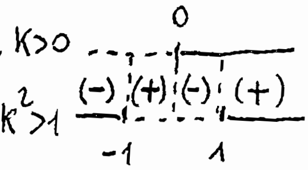
$2=2\cdot(\cos 0+i\,\mathrm{sen}\,0)$;

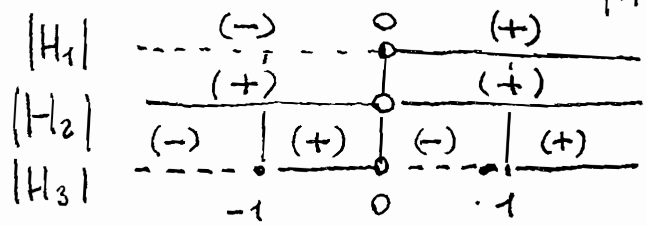
$\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i = 1\cdot\left(\cos\frac{\pi}{3}+i\,\mathrm{sen}\frac{\pi}{3}\right)$; $\frac{1}{2}-\frac{\sqrt{3}}{2}i = 1\cdot\left(\cos\frac{5}{3}\pi+i\,\mathrm{sen}\frac{5}{3}\pi\right)$.

$$2\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2\cdot1\cdot1\cdot\left(\cos\left(0+\frac{\pi}{3}+\frac{5}{3}\pi\right)+i\,\mathrm{sen}\left(0+\frac{\pi}{3}+\frac{5}{3}\pi\right)\right)=2\left(\cos 2\pi+i\,\mathrm{sen}\,2\pi\right)=2.$$

IM2)

$$H=\begin{Vmatrix} K & 0 & 1 \\ 0 & K & 0 \\ 1 & 0 & K \end{Vmatrix} \Rightarrow \begin{cases} |H_1| = K>0; \\ |H_2|=\begin{vmatrix} K & 0 \\ 0 & K\end{vmatrix} = K^2>0 \text{ per } K\neq 0; \\ |H_3|=\begin{vmatrix} K & 0 & 1 \\ 0 & K & 0 \\ 1 & 0 & K \end{vmatrix} = K\begin{vmatrix} K & 1 \\ 1 & K\end{vmatrix} = K\cdot(K^2-1)>0 \left\langle \begin{array}{l} K>0 \\ K^2>1 \end{array}\right. \end{cases}$$





$\Rightarrow \begin{cases} \text{Se } K<-1: \text{ definita negativa} \\ \text{Se } -1<K<0: \text{ indefinita} \\ \text{Se } 0<K<1: \text{ indefinita} \\ \text{Se } K>1: \text{ definita positiva.} \end{cases}$

IM3) $f(x;y)=x^2+y^2+6xy+1=0$; $f\left(\frac{1}{2};-\frac{1}{2}\right)=\frac{1}{4}+\frac{1}{4}-\frac{6}{4}+1=0$.

$\nabla f(x;y)=(2x+6y;\,2y+6x)$; $\nabla f\left(\frac{1}{2};-\frac{1}{2}\right)=(1-3;\,-1+3)=(-2;2)$.

$H(f(x;y))=\begin{Vmatrix} 2 & 6 \\ 6 & 2 \end{Vmatrix}=H\left(f\left(\frac{1}{2};-\frac{1}{2}\right)\right)$. $\quad y=y(x)$ in $x=\frac{1}{2}$.

$y'\left(\frac{1}{2}\right)=-\frac{-2}{2}=1$; $y''\left(\frac{1}{2}\right)=-\frac{2+2\cdot6\cdot1+2\cdot1}{2}=-8$.

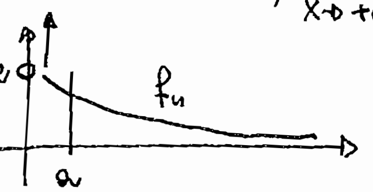
$P_2\left(x;\frac{1}{2}\right)=-\frac{1}{2}+1\cdot\left(x-\frac{1}{2}\right)+(-8)\cdot\frac{1}{2}\left(x-\frac{1}{2}\right)^2=-2+5x-4x^2.$

IM4) $f_n(x)=n\cdot e^{1-nx}$.

$$\lim_{n\to+\infty} f_n(x)=\lim_{n\to+\infty} n\cdot e^{1-nx}=\begin{cases} 0: & x>0 \\ +\infty: & x=0 \\ +\infty: & x<0 \end{cases} \Rightarrow C=]0;+\infty[;\ f(x)=0\ \forall x\in C.$$

Studiamo $f_n(x)$.

$f_n(0)=n\cdot e$; $\lim_{x\to+\infty} f_n(x)=0$; $f_n'(x)=n\cdot e^{1-nx}\cdot(-n)<0\ \forall x\in C$.



Preso $a>0$: $\sup_{x\in[a;+\infty[}\{|f_n(x)-0|\}=f_n(a)=ne^{1-na}$ e dato che $\lim_{n\to+\infty} n\cdot e^{1-na}=0$ si ha C.U. in ogni $[a;+\infty[$ con $a>0$.

$\sum_{0}^{+\infty} n\, e^{1-nx}$. Usiamo il criterio della Radice: AM2

$$\lim_{n\to+\infty} \sqrt[n]{|f_n(x)|} = \lim_{n\to+\infty} \sqrt[n]{n}\cdot\sqrt[n]{e}\cdot\sqrt[n]{(e^{-x})^n} = 1\cdot 1\cdot e^{-x} = e^{-x} < 1 \text{ per } x>0.$$

La Serie di funzioni è convergente $\forall x>0$: $C =]0;+\infty[$.

II M1) $f(x;y) = y^2 e^x - xy$.

$$\begin{cases} f'_x = y^2e^x - y = y(ye^x-1) = 0 \\ f'_y = 2ye^x - x = 0 \end{cases} \Rightarrow \begin{cases} y=0 \\ x=0 \end{cases} \text{ oppure } \begin{cases} y = e^{-x} \\ 2e^{-x}\cdot e^x - x = 2-x = 0 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=e^{-2}. \end{cases}$$

$P_1 = (0;0)$; $P_2 = (2;e^{-2})$. $H(x;y) = \begin{Vmatrix} y^2e^x & 2ye^x-1 \\ 2ye^x-1 & 2e^x \end{Vmatrix}$.

$H(0;0) = \begin{Vmatrix} 0 & -1 \\ -1 & 2 \end{Vmatrix}$; $|H| = -1<0$: Sella; $H(2;e^{-2}) = \begin{Vmatrix} e^{-2} & 1 \\ 1 & 2e^2 \end{Vmatrix} \Rightarrow \begin{cases} e^{-2}>0; 2e^2>0 \\ 2-1=1>0 \end{cases}$: Minimo.

II M2) $\begin{cases} \text{Max/min } f(x;y;z) = x^2-y+z^2 \\ \text{s.v.: } x-y^2+z=0. \end{cases}$

Dal vincolo otteniamo: $z = y^2-x$ da cui:

$f(x;y) = x^2-y+(y^2-x)^2 = x^2-y+y^4+x^2-2xy^2 = 2x^2+y^4-2xy^2-y$.

Cerchiamo Max e min liberi di questa funzione.

$$\begin{cases} f'_x = 4x-2y^2 = 2(2x-y^2) = 0 \\ f'_y = 4y^3-4xy-1 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{y^2}{2} \\ 4y^3 - 4\frac{y^2}{2}\cdot y - 1 = 2y^3-1 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\sqrt[3]{4}} \\ y = \frac{1}{\sqrt[3]{2}}. \end{cases}$$

$$H(x;y) = \begin{Vmatrix} 4 & -4y \\ -4y & 12y^2-4x \end{Vmatrix}; H\left(\frac{1}{2\sqrt[3]{4}}; \frac{1}{\sqrt[3]{2}}\right) = \begin{Vmatrix} 4 & -\frac{4}{\sqrt[3]{2}} \\ -\frac{4}{\sqrt[3]{2}} & \frac{12}{\sqrt[3]{4}} - \frac{2}{\sqrt[3]{4}} \end{Vmatrix} = \begin{Vmatrix} 4 & -\frac{4}{\sqrt[3]{2}} \\ -\frac{4}{\sqrt[3]{2}} & \frac{10}{\sqrt[3]{4}} \end{Vmatrix}$$

$|H_1| = 4>0$; $|H_1| = \frac{10}{\sqrt[3]{4}} > 0$; $|H_2| = \frac{40}{\sqrt[3]{4}} - \frac{16}{\sqrt[3]{4}} = \frac{24}{\sqrt[3]{4}} > 0$: Punto di minimo.

Se $x = \frac{1}{2\sqrt[3]{4}}$ e $y = \frac{1}{\sqrt[3]{2}} \Rightarrow z = y^2 - x = \frac{1}{\sqrt[3]{4}} - \frac{1}{2\sqrt[3]{4}} = \frac{1}{2\sqrt[3]{4}}$.

II M3) $\begin{cases} xy' = (1+y^2)\cdot \log x \\ y(1) = 1 \end{cases}$

AM 3

$$x\cdot y' = (1+y^2)\cdot \log x \Rightarrow \frac{1}{1+y^2}\cdot y' = \frac{1}{x}\cdot \log x \Rightarrow$$

$$\Rightarrow \int \frac{1}{1+y^2}\,dy = \int \frac{1}{x}\log x\,dx + k = \int \log x\,d(\log x) + k \Rightarrow$$

$$\Rightarrow \operatorname{arctg} y = \frac{1}{2}\log^2 x + k \Rightarrow y = \operatorname{tg}\left(\frac{1}{2}\log^2 x + k\right).$$

$$y(1) = \operatorname{tg}\left(\frac{1}{2}\cdot 0 + k\right) = \operatorname{tg} k = 1 \Rightarrow k = \operatorname{arctg} 1 = \frac{\pi}{4}.$$

Soluzione: $y = \operatorname{tg}\left(\frac{1}{2}\log^2 x + \frac{\pi}{4}\right)$.

II M4) $f(x;y) = e^{x^2-y^2}$. $f(0;0) = 1$.

$$\nabla f(x;y) = \left(2x e^{x^2-y^2}; -2y e^{x^2-y^2}\right); \nabla f(0;0) = (0;0)$$

$$H(f(x;y)) = \begin{Vmatrix} 2e^{x^2-y^2} + 4x^2 e^{x^2-y^2} & -4xy\, e^{x^2-y^2} \\ -4xy\, e^{x^2-y^2} & -2e^{x^2-y^2} + 4y^2 e^{x^2-y^2} \end{Vmatrix}$$

$$H(f(0;0)) = \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}$$

$$P_2((x;y);(0;0)) = f(0;0) + \nabla f(0;0)\cdot(x-0;y-0) + \frac{1}{2}\begin{Vmatrix} x-0 & y-0 \end{Vmatrix}\cdot H(f(0;0))\cdot \begin{Vmatrix} x-0 \\ y-0 \end{Vmatrix} =$$

$$P_2(x;y) = 1 + (0;0)\cdot(x;y) + \frac{1}{2}\begin{Vmatrix} x & y \end{Vmatrix}\cdot \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}\cdot \begin{Vmatrix} x \\ y \end{Vmatrix} =$$

$$P_2(x;y) = 1 + \frac{1}{2}(2x^2 - 2y^2) = 1 + x^2 - y^2.$$