Compito di Analisi Matematica del 19/1/2015 $\boxed{CAM1}$

IM1) $a+b=2$; $a\cdot b=4 \Rightarrow x^2-2x+4=0 \Rightarrow x=1\pm\sqrt{1-4} = 1\pm\sqrt{3}\cdot i$.

•) $1+\sqrt{3}\cdot i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot i\right) = 2\cdot\left(\cos\frac{\pi}{3}+i\,\text{sen}\,\frac{\pi}{3}\right)$.

$\sqrt{1+\sqrt{3}\cdot i} = \sqrt{2}\cdot\left(\cos\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{2}\right)+i\,\text{sen}\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{2}\right)\right); 0\le k\le 1.$

$k=0$: $\sqrt{2}\cdot\left(\cos\frac{\pi}{6}+i\,\text{sen}\,\frac{\pi}{6}\right)=\sqrt{2}\left(\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)$; $k=1$: $\sqrt{2}\left(\cos\frac{7}{6}\pi+i\,\text{sen}\,\frac{7}{6}\pi\right)=\sqrt{2}\cdot\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)$.

••) $1-\sqrt{3}\cdot i = 2\cdot\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 2\cdot\left(\cos\frac{5}{3}\pi+i\,\text{sen}\,\frac{5}{3}\pi\right)$

$\sqrt{1-\sqrt{3}\cdot i} = \sqrt{2}\cdot\left(\cos\left(\frac{5}{6}\pi+k\cdot\frac{2\pi}{2}\right)+i\,\text{sen}\left(\frac{5}{6}\pi+k\cdot\frac{2\pi}{2}\right)\right);\ 0\le k\le 1.$

$k=0$: $\sqrt{2}\cdot\left(\cos\frac{5}{6}\pi+i\,\text{sen}\,\frac{5}{6}\pi\right)=\sqrt{2}\cdot\left(-\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)$; $k=1$: $\sqrt{2}\cdot\left(\cos\frac{11}{6}\pi+i\,\text{sen}\,\frac{11}{6}\pi\right)=\sqrt{2}\cdot\left(\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}\right)$.

IM2) $f(x;y)=|xy|\cdot(x-y)$ è continua $\forall (x;y)\in\mathbb{R}^2$. $f(0;0)=0$.

$$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to 0}\frac{|(0+h)\cdot 0|\cdot(0+h-0)-0}{h}=\lim_{h\to 0}\frac{|0|\cdot h-0}{h}=\lim_{h\to 0}\frac{0}{h}=0;$$

$$\frac{\partial f}{\partial y}(0;0)=\lim_{h\to 0}\frac{|0\cdot(0+h)|\cdot(0-(0+h))-0}{h}=\lim_{h\to 0}\frac{|0|\cdot h-0}{h}=\lim_{h\to 0}\frac{0}{h}=0.$$

$f(x;y)$ è differenziabile in $(0;0)$ se $\lim\limits_{(x;y)\to(0;0)}\dfrac{|xy|\cdot(x-y)-0-(0;0)(x-0;y-0)}{\sqrt{x^2+y^2}}=0 \Rightarrow$

$$\Rightarrow \lim_{\rho\to 0}\frac{\rho^2\cdot|\cos\vartheta\,\text{sen}\,\vartheta|\cdot\rho(\cos\vartheta-\text{sen}\,\vartheta)}{\rho}=\lim_{\rho\to 0}\rho^2\cdot|\cos\vartheta\,\text{sen}\,\vartheta|\cdot(\cos\vartheta-\text{sen}\,\vartheta)=0.$$

La convergenza è uniforme in quanto $\left|\rho^2|\cos\vartheta\,\text{sen}\,\vartheta|\cdot(\cos\vartheta-\text{sen}\,\vartheta)\right|\le\rho^2\cdot 1\cdot 2=2\rho^2$.

Quindi la funzione è differenziabile in $(0;0)$.

IM3) $f(x;y)=e^{x-y}$ è funzione differenziabile due volte $\forall (x;y)\in\mathbb{R}^2$.

$\nabla f=\left(e^{x-y};-e^{x-y}\right)$; $H(f)=\begin{Vmatrix} e^{x-y} & -e^{x-y}\\ -e^{x-y} & e^{x-y}\end{Vmatrix}$.

$D_u f(x;y)=\nabla f(x;y)\cdot u=\left(e^{x-y};-e^{x-y}\right)\cdot\left(\frac{\sqrt{3}}{2};\frac{1}{2}\right)=\frac{1}{2}e^{x-y}\cdot(\sqrt{3}-1)$.

$$D^2_{u,v}f(x;y)=u\cdot H(f(x;y))\cdot v^T=\begin{Vmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{Vmatrix}\cdot\begin{Vmatrix} e^{x-y} & -e^{x-y}\\ -e^{x-y} & e^{x-y}\end{Vmatrix}\cdot\begin{Vmatrix}-\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{Vmatrix}=\begin{Vmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{Vmatrix}\cdot\begin{Vmatrix} e^{x-y}\cdot\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\\ e^{x-y}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\end{Vmatrix}=$$

$= e^{x-y}\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}\right)+e^{x-y}\left(\frac{\sqrt{3}}{4}+\frac{1}{4}\right)= e^{x-y}\cdot\left(-\frac{1}{2}\right) = -2$ se $e^{x-y}=4$.

Quindi $D_u f(x;y) = \frac{1}{2}e^{x-y}\cdot(\sqrt{3}-1) = \frac{1}{2}\cdot 4\cdot(\sqrt{3}-1) = 2(\sqrt{3}-1)$.

I M4) $\sum_{n=0}^{+\infty}\frac{n}{n^2+3n-1}\cdot x^n$. Centro della serie: $x=0$.

$$\rho = \lim_{n\to+\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to+\infty}\frac{n}{n^2+3n-1}\cdot\frac{(n+1)^2+3(n+1)-1}{n+1} = \lim_{n\to+\infty}\frac{n^3}{n^3} = 1 \Rightarrow \rho = 1$$

La serie di potenze converge almeno in $]-1;+1[$.

Per $x=-1$: $\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{n}{n^2+3n-1}$: Serie a segni alterni, Convergente per Cr. Leibnitz;

Per $x=1$: $\sum_{n=0}^{+\infty}\frac{n}{n^2+3n-1}$: Serie Divergente in quanto $\frac{n}{n^2+3n-1}\sim\frac{1}{n}$.

Quindi $C=[-1,1[$.

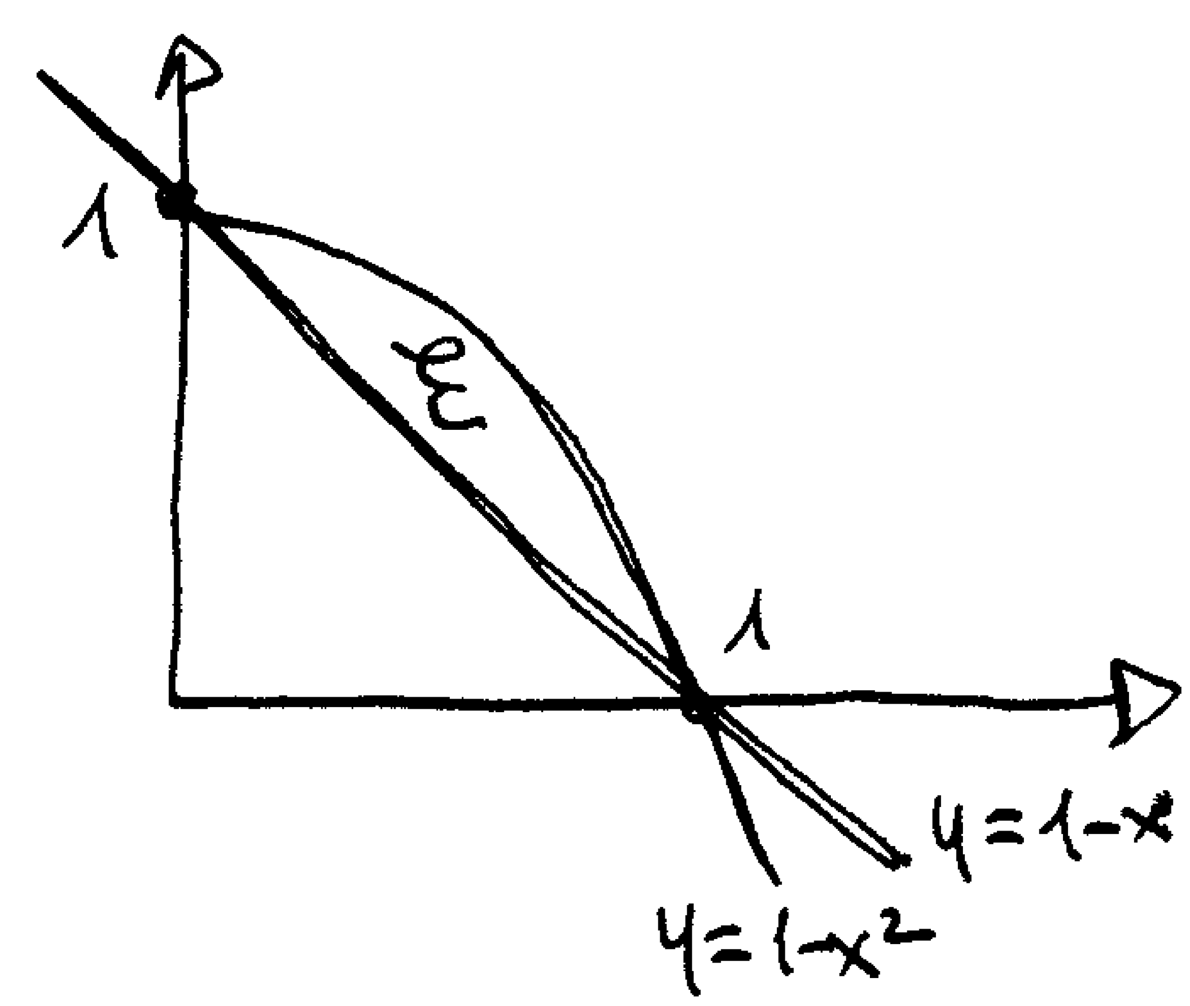
II M1) $\begin{cases} f(x;y;z) = xy-xz+y=0 \\ g(x;y;z) = xyz-xz+yz=0 \end{cases}$; $P=(-1;1;0) \Rightarrow \begin{cases} f(-1;1;0) = -1-0+1=0 \\ g(-1;1;0)=0-0+0=0 \end{cases}$.

$$\frac{\partial(f;g)}{\partial(x;y;z)} = \left\|\begin{matrix} y-z & x+1 & -x \\ yz-z & xz+z & xy-x+y \end{matrix}\right\|;\ \frac{\partial(f;g)}{\partial(x;y;z)}(-1;1;0) = \left\|\begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{matrix}\right\|$$

Possiamo definire una funzione implicita $y\to(x(y);z(y))$ dato che $\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1\neq 0$.

$\frac{dx}{dy} = -\frac{\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}} = 0$; $\frac{dz}{dy} = -\frac{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}} = 0$. Quindi non esiste il vettore tangente in $y=1$.

II M2) $\begin{cases} \text{Max/min } f(x;y) = x^2-xy \\ \text{s.v.} \begin{cases} 1-x\leq y \\ y\leq 1-x^2 \end{cases} \end{cases} \Rightarrow \begin{cases} \text{Max/min } f(x;y) = x^2-xy \\ \text{s.v.} \begin{cases} 1-x-y\leq 0 \\ y+x^2-1\leq 0 \end{cases} \end{cases}$.



$f(x;y)$ è una funzione continua. E è un insieme limitato e chiuso. I vincoli sono qualificati.

Per il Teorema di Weierstrass esistono il Max ed il min.

CAM3

$$\Lambda = x^2 - xy - \lambda_1(1-x-y) - \lambda_2(y+x^2-1)$$

Caso $\lambda_1 = \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = 2x - y = 0 \\ \Lambda'_y = -x = 0 \\ y \geq 1-x \\ y \leq 1-x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \geq 1 \\ 0 \leq 1 \end{cases} \Rightarrow (0;0) \notin \mathcal{E}.$$

Caso $\lambda_1 \neq 0;\ \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = 2x - y + \lambda_1 = 0 \\ \Lambda'_y = -x + \lambda_1 = 0 \\ y = 1-x \\ y \leq 1-x^2 \end{cases} \Rightarrow \begin{cases} x = \lambda_1 \\ y = 1-\lambda_1 \\ 2\lambda_1 - 1 + \lambda_1 + \lambda_1 = 4\lambda_1 - 1 = 0 \\ y \leq 1-x^2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{1}{4} > 0 \\ x = \frac{1}{4} \\ y = \frac{3}{4} \\ \frac{3}{4} \leq 1 - \frac{1}{16} : \text{Vera} \end{cases} \Rightarrow \left(\tfrac{1}{4}; \tfrac{3}{4}\right) \text{ Max ??}$$

Caso $\lambda_1 = 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2x - y - 2\lambda_2 x = 0 \\ \Lambda'_y = -x - \lambda_2 = 0 \\ y = 1-x^2 \\ y \geq 1-x \end{cases} \Rightarrow \begin{cases} x = -\lambda_2 \\ y = 1-\lambda_2^2 \\ -2\lambda_2 - 1 + \lambda_2^2 + 2\lambda_2^2 = 3\lambda_2^2 - 2\lambda_2 - 1 = 0 \\ y \geq 1-x \end{cases}$$

$$\lambda_2 = \frac{1 \pm \sqrt{1+3}}{3} = \frac{1 \pm 2}{3}$$

$$\Rightarrow \begin{cases} \lambda_2 = 1 \\ x = -1 \\ y = 0 \\ 0 \geq 2 \end{cases} \cup \begin{cases} \lambda_2 = -\frac{1}{3} < 0 \\ x = \frac{1}{3} \\ y = \frac{8}{9} \\ \frac{8}{9} \geq \frac{2}{3} : \text{vera} \end{cases}$$

$(-1;0) \notin \mathcal{E}$ $\qquad$ $\left(\frac{1}{3}; \frac{8}{9}\right)$ min ??

Caso $\lambda_1 \neq 0;\ \lambda_2 \neq 0$

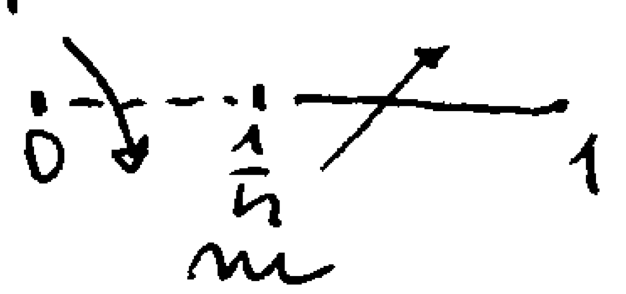
$$\begin{cases} \Lambda'_x = 2x - y + \lambda_1 - 2\lambda_2 x = 0 \\ \Lambda'_y = -x + \lambda_1 - \lambda_2 = 0 \\ \begin{cases} y = 1-x \\ y = 1-x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \cup \begin{cases} x = 1 \\ y = 0 \end{cases} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = 1 \\ \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = 1 > 0 \\ \lambda_2 = 1 > 0 \end{cases}$$

$(0;1)$ MAX ??

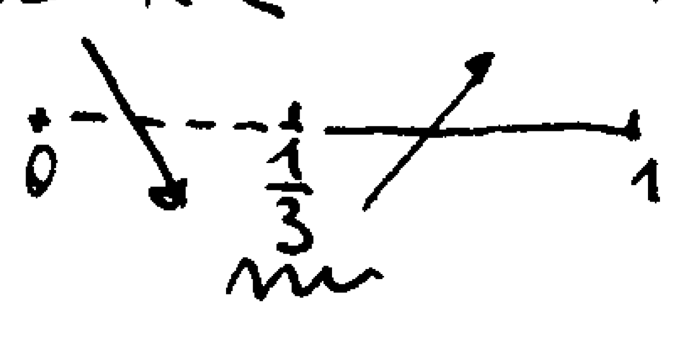
$$\cup \begin{cases} x = 1 \\ y = 0 \\ \lambda_1 - 2\lambda_2 = -2 \\ \lambda_1 - \lambda_2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_1 = 4 > 0 \\ \lambda_2 = 3 > 0 \end{cases}$$

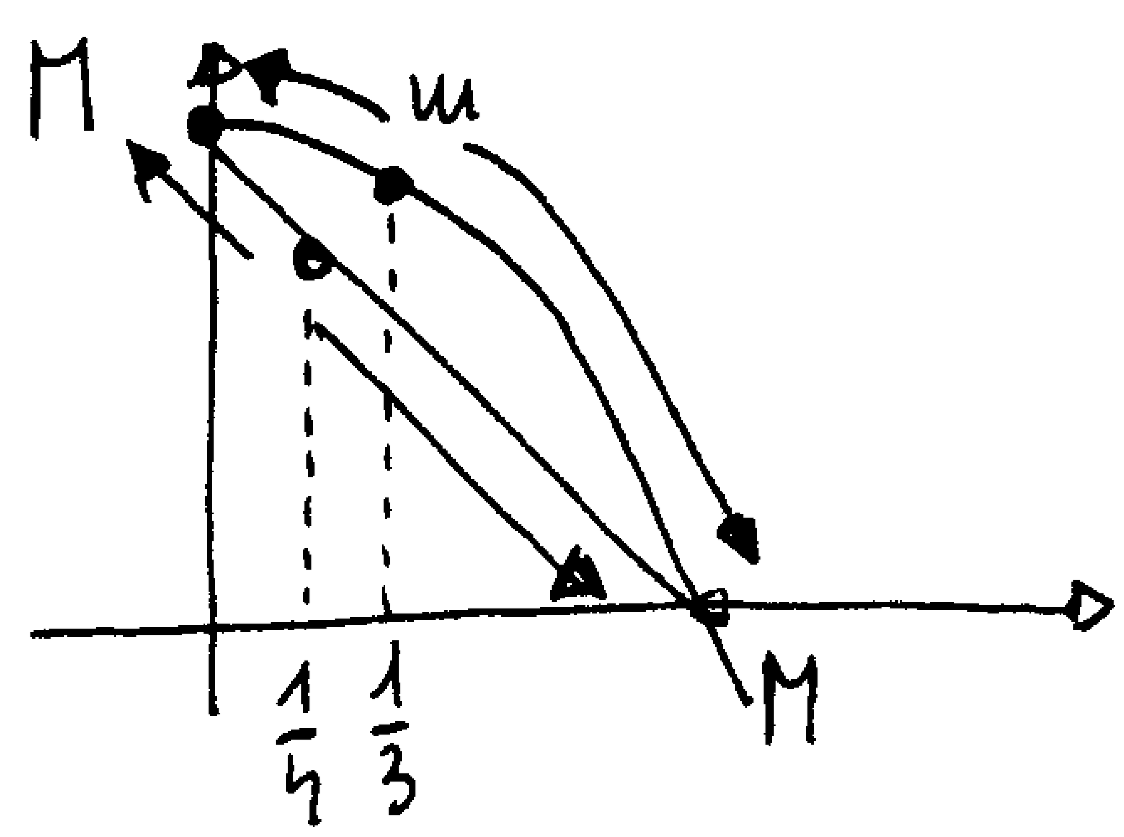
$(1;0)$ MAX ??

$f(x; 1-x) = 2x^2 - x;\ f'(x) = 4x - 1 \geq 0 : x \geq \frac{1}{4}$



$f(x; 1-x^2) = x^3 + x^2 - x;\ f'(x) = 3x^2 - 2x - 1 \geq 0$ per $x \leq -1 \cup x \geq \frac{1}{3}$





$\left(\frac{1}{3};\frac{8}{9}\right)$ è il punto di minimo; $f\left(\frac{1}{3};\frac{8}{9}\right)=-\frac{5}{27}$.

$(0;1)$ e $(1;0)$ sono punti di massimo; $f(0;1)=0$; $f(1;0)=1$ (Max assoluto)

Il punto $\left(\frac{1}{4};\frac{3}{4}\right)$ non è né punto di Massimo né punto di minimo.

II M3) $\begin{cases} y'\cdot\log^2 y = xy \\ y(0)=e \end{cases} \Rightarrow \int \frac{\log^2 y}{y}\,dy = \int x\,dx + K \Rightarrow \frac{1}{3}\log^3 y = \frac{x^2}{2}+k \Rightarrow$

$\Rightarrow \log^3 y = \frac{3}{2}x^2 + m \quad (m=3k) \Rightarrow \log y = \sqrt[3]{\frac{3}{2}x^2+m} \Rightarrow y = e^{\sqrt[3]{\frac{3}{2}x^2+m}}$.

$y(0) = e^{\sqrt[3]{m}} = e \Rightarrow m = 1 \quad \left(k=\frac{1}{3}\right)$. Soluzione: $y = e^{\sqrt[3]{\frac{3}{2}x^2+1}}$.

II M4) $\begin{cases} x' = y+t \\ y' = x+e^t \end{cases} \Rightarrow x'' = y'+1 = x+e^t+1 \Rightarrow x''-x = e^t+1.$

$(D^2-1)(x)=0 \Rightarrow (D-1)(D+1)(x)=0 \Rightarrow x(t) = c_1 e^t + c_2 e^{-t}$.

$x_0 = ae^t + bte^t + c$ (dato che $D-1$ annichila e^t)

$x_0' = (a+b)e^t + bte^t$; $x_0'' = (a+2b)e^t + bte^t$. Sostituendo:

$(\cancel{a}+2b)e^t + \cancel{bte^t} - \cancel{ae^t} - \cancel{bte^t} - c = e^t+1 \Rightarrow$

$\Rightarrow \begin{cases} 2b=1 \\ -c=1 \end{cases} \Rightarrow \begin{cases} b=\frac{1}{2} \\ c=-1 \end{cases} \Rightarrow x(t) = c_1e^t + c_2e^{-t} + \frac{1}{2}te^t - 1$;

$y(t) = x'(t) - t = c_1e^t - c_2e^{-t} + \frac{1}{2}e^t + \frac{1}{2}te^t - t.$