Prova Intermedia di Analisi Matematica del 14/12/2014 $\boxed{\text{AM1}}$

IM1) $\left(x-\frac{1}{\sqrt{2}}\right)^4=-1 \Rightarrow x-\frac{1}{\sqrt{2}}=\sqrt[4]{-1}$. $\quad -1=\cos\pi+i\,\mathrm{sen}\,\pi$.

$\sqrt[4]{-1}=\sqrt[4]{1}\cdot\left(\cos\left(\frac{\pi}{4}+k\cdot\frac{2\pi}{4}\right)+i\,\mathrm{sen}\left(\frac{\pi}{4}+k\cdot\frac{2\pi}{4}\right)\right);\ 0\leq k\leq 3.$

$\underline{k=0}$: $\cos\frac{\pi}{4}+i\,\mathrm{sen}\,\frac{\pi}{4}=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$; $\underline{k=1}$: $\cos\frac{3}{4}\pi+i\,\mathrm{sen}\,\frac{3}{4}\pi=-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$;

$\underline{k=2}$: $\cos\frac{5}{4}\pi+i\,\mathrm{sen}\,\frac{5}{4}\pi=-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$; $\underline{k=3}$: $\cos\frac{7}{4}\pi+i\,\mathrm{sen}\,\frac{7}{4}\pi=\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$.

$x_0=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}+i\frac{1}{\sqrt{2}}$; $x_1=-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=i\frac{1}{\sqrt{2}}$;

$x_2=-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=-i\frac{1}{\sqrt{2}}$; $x_3=\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}-i\cdot\frac{1}{\sqrt{2}}$.

IM2) $f(x;y)=\begin{cases} x\cdot\frac{x^2-y^2}{x^2+y^2} & : (x;y)\neq(0;0) \\ 0 & : (x;y)=(0;0)\end{cases}$. $\lim\limits_{(x;y)\to(0;0)} x\cdot\frac{x^2-y^2}{x^2+y^2} \Rightarrow$ Coordinate polari

$$\Rightarrow \lim_{\rho\to0}\rho\cdot\cos\vartheta\cdot\frac{\rho^2\cos^2\vartheta-\rho^2\mathrm{sen}^2\vartheta}{\rho^2\cos^2\vartheta+\rho^2\mathrm{sen}^2\vartheta}=\lim_{\rho\to0}\rho\cos\vartheta\cdot\frac{\rho^2\cos2\vartheta}{\rho^2}=$$

$=\lim\limits_{\rho\to0}\rho\cdot\cos\vartheta\cdot\cos2\vartheta=0$. La convergenza è uniforme rispetto a ϑ:

$|\rho\cdot\cos\vartheta\cdot\cos2\vartheta-0|\leq\rho\cdot1\cdot1<\varepsilon$. quindi $f(x;y)\in\mathcal{C}(0;0)$.

Per la differenziabilità: $\lim\limits_{(x;y)\to(0;0)}\frac{f(x;y)-f(0;0)-\nabla f(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}\overset{?}{=}0$.

$$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to0}\left[(0+h)\cdot\frac{(0+h)^2-0^2}{(0+h)^2+0^2}-0\right]\cdot\frac{1}{h}=\lim_{h\to0}h\cdot\frac{h^2}{h^2}\cdot\frac{1}{h}=1;$$

$$\frac{\partial f}{\partial y}(0;0)=\lim_{h\to0}\left[0\cdot\frac{0^2-(0+h)^2}{0^2+(0+h)^2}-0\right]\cdot\frac{1}{h}=\lim_{h\to0}0=0.$$

$$\lim_{(x;y)\to(0;0)}\left[x\cdot\frac{x^2-y^2}{x^2+y^2}-0-(1;0)(x;y)\right]\cdot\frac{1}{\sqrt{x^2+y^2}}=\lim_{(x;y)\to(0;0)}\left(x\cdot\frac{x^2-y^2}{x^2+y^2}-x\right)\cdot\frac{1}{\sqrt{x^2+y^2}}=$$

$=\lim\limits_{(x;y)\to(0;0)}\frac{x^3-xy^2-x^3-xy^2}{(x^2+y^2)\sqrt{x^2+y^2}}=\lim\limits_{(x;y)\to(0;0)}\frac{-2xy^2}{(x^2+y^2)\sqrt{x^2+y^2}}\Rightarrow$ Coordinate polari

$$\Rightarrow\lim_{\rho\to0}\frac{-2\rho^3\cos\vartheta\,\mathrm{sen}^2\vartheta}{\rho^2\cdot\rho}=\lim_{\rho\to0}(-2\cos\vartheta\,\mathrm{sen}^2\vartheta)=-2\cos\vartheta\,\mathrm{sen}^2\vartheta\neq0.$$

La funzione $f(x;y)$ non è differenziabile in $(0;0)$.

AM2

IM3) $f(x;y)=\begin{cases} x\cdot\frac{x^2-y^2}{x^2+y^2} & :(x;y)\neq(0;0)\\ 0 & :(x;y)=(0;0)\end{cases}$. $\mathcal{D}_v f(0;0)=\lim_{t\to0}\frac{f((0;0)+tv)-f(0;0)}{t}\Rightarrow$

$\Rightarrow v=(\cos\alpha;\operatorname{sen}\alpha):\lim_{t\to0}\frac{f(t\cos\alpha;t\operatorname{sen}\alpha)-0}{t}=\lim_{t\to0}\left[t\cos\alpha\cdot\frac{t^2\cos^2\alpha-t^2\operatorname{sen}^2\alpha}{t^2\cos^2\alpha+t^2\operatorname{sen}^2\alpha}-0\right]\cdot\frac{1}{t}=$

$=\lim_{t\to0} t\cdot\cos\alpha\cdot\frac{t^2\cos2\alpha}{t^2}\cdot\frac{1}{t}=\lim_{t\to0}\cos\alpha\cdot\cos2\alpha=\cos\alpha\cdot\cos2\alpha.$

Si noti che $\mathcal{D}_v f(0;0)=\cos\alpha\cdot\cos2\alpha\neq\nabla f(0;0)\cdot(\cos\alpha;\operatorname{sen}\alpha)=(1;0)(\cos\alpha;\operatorname{sen}\alpha)=\cos\alpha.$

IM4) $f(x;y)=e^{2x-y}$: funzione differenziabile due volte $\forall(x;y)\in\mathbb{R}^2$.

$\nabla f(x;y)=(2e^{2x-y};-e^{2x-y})$. $\mathcal{D}_v f(x;y)=\nabla f(x;y)\cdot(\cos\alpha;\operatorname{sen}\alpha)=(\alpha=\frac{\pi}{4})=$

$=(2e^{2x-y};-e^{2x-y})\cdot(\cos\frac{\pi}{4};\operatorname{sen}\frac{\pi}{4})=e^{2x-y}\cdot(2;-1)\cdot(\frac{1}{\sqrt2};\frac{1}{\sqrt2})=\frac{1}{\sqrt2}e^{2x-y}.$

$\mathcal{D}^2_{v,w}f(x;y)=v\cdot H(x;y)\cdot w^T=\begin{Vmatrix}\cos\alpha & \operatorname{sen}\alpha\end{Vmatrix}\cdot\begin{Vmatrix}4e^{2x-y} & -2e^{2x-y}\\ -2e^{2x-y} & e^{2x-y}\end{Vmatrix}\cdot\begin{Vmatrix}\cos\beta\\ \operatorname{sen}\beta\end{Vmatrix}=(\beta=\frac{3}{4}\pi)=$

$=\begin{Vmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{Vmatrix}\begin{Vmatrix}4e^{2x-y} & -2e^{2x-y}\\ -2e^{2x-y} & e^{2x-y}\end{Vmatrix}\begin{Vmatrix}-\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{Vmatrix}=\frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2}\cdot e^{2x-y}\cdot\begin{Vmatrix}1 & 1\end{Vmatrix}\cdot\begin{Vmatrix}4 & -2\\ -2 & 1\end{Vmatrix}\cdot\begin{Vmatrix}-1\\ 1\end{Vmatrix}=$

$=\frac{1}{2}e^{2x-y}\cdot\begin{Vmatrix}1 & 1\end{Vmatrix}\cdot\begin{Vmatrix}-6\\ 3\end{Vmatrix}=\frac{1}{2}e^{2x-y}\cdot(-6+3)=-\frac{3}{2}e^{2x-y}=-6\Rightarrow e^{2x-y}=4.$

Quindi $\mathcal{D}_v f(x;y)=\frac{1}{\sqrt2}\cdot e^{2x-y}=\frac{1}{\sqrt2}\cdot4=2\sqrt2.$

IM5) $\begin{cases} f(x;y;z;w)=xy+zw-e^{y-z}-e^{x-w}=0\\ g(x;y;z;w)=e^{x-y}+e^{z-w}-xz-yw=0\end{cases}$ $P_0=(1;1;1;1)$.

$\begin{cases} f(P_0)=1+1-e^0-e^0=0\\ g(P_0)=e^0+e^0-1-1=0\end{cases}$. f e g funzioni differenziabili.

$\frac{\partial(f;g)}{\partial(x;y;z;w)}=\begin{Vmatrix}y-e^{x-w} & x-e^{y-z} & w+e^{y-z} & z+e^{x-w}\\ e^{x-y}-z & -e^{x-y}-w & e^{z-w}-x & -e^{z-w}-y\end{Vmatrix}.$

$\frac{\partial(f;g)}{\partial(x;y;z;w)}(1;1;1;1)=\begin{Vmatrix}1-1 & 1-1 & 1+1 & 1+1\\ 1-1 & -1-1 & 1-1 & -1-1\end{Vmatrix}=\begin{array}{c}\begin{Vmatrix}0 & 0 & 2 & 2\\ 0 & -2 & 0 & -2\end{Vmatrix}\\ \;\;x\;\;\;\;y\;\;\;\;z\;\;\;\;w\end{array}.$

Dato che $\begin{array}{c} \begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix} \\ z \quad w \end{array} = -4-0 \neq 0$ si può definire una funzione implicita

$F: (x;y) \to (z;w)$.

$$J_F = \frac{\partial(z;w)}{\partial(x;y)} = \begin{Vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{Vmatrix} =$$

$$= \begin{Vmatrix} -\dfrac{\begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix}} & -\dfrac{\begin{vmatrix} 0 & 2 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix}} \\ -\dfrac{\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix}} & -\dfrac{\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 0 & -2 \end{vmatrix}} \end{Vmatrix} = \begin{Vmatrix} -\frac{0}{-4} & -\frac{4}{-4} \\ -\frac{0}{-4} & -\frac{-4}{-4} \end{Vmatrix} = \begin{Vmatrix} 0 & 1 \\ 0 & -1 \end{Vmatrix}.$$