Compito di Analisi Matematica del 27/1/2016 $\boxed{AM1}$

IM1) $1+i = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\cdot\left(\cos\frac{\pi}{4}+i\,\text{sen}\frac{\pi}{4}\right)$;

$\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) = 2\left(\cos\frac{\pi}{6}+i\,\text{sen}\frac{\pi}{6}\right)$.

$$\frac{1+i}{\sqrt{3}+i} = \frac{\sqrt{2}}{2}\cdot\frac{\cos\frac{\pi}{4}+i\,\text{sen}\frac{\pi}{4}}{\cos\frac{\pi}{6}+i\,\text{sen}\frac{\pi}{6}} = \frac{1}{\sqrt{2}}\cdot\left(\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)+i\,\text{sen}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right) = \frac{1}{\sqrt{2}}\cdot\left(\cos\frac{\pi}{12}+i\,\text{sen}\frac{\pi}{12}\right).$$

$$\frac{1+i}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{(\sqrt{3}+1)+i(\sqrt{3}-1)}{4} = \frac{1}{\sqrt{2}}\cdot\left(\frac{\sqrt{3}+1}{2\sqrt{2}}+i\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \Rightarrow \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}};\ \text{sen}\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

IM2) $f(x;y) = \begin{cases} x^2\cdot\frac{x-y}{x^2+y^2} & : (x;y)\neq(0;0) \\ 0 & : (x;y)=(0;0) \end{cases}$.

$$\lim_{(x;y)\to(0;0)} x^2\cdot\frac{x-y}{x^2+y^2} \Rightarrow \lim_{\rho\to0}\rho^2\cos^2\vartheta\cdot\frac{\rho(\cos\vartheta-\text{sen}\vartheta)}{\rho^2} = \lim_{\rho\to0}\rho\cdot\cos^2\vartheta\cdot(\cos\vartheta-\text{sen}\vartheta) = 0.$$

La convergenza è uniforme: $|\rho\cdot\cos^2\vartheta\cdot(\cos\vartheta-\text{sen}\vartheta)| \leq \rho\cdot|\cos^2\vartheta|\cdot(|\cos\vartheta|+|\text{sen}\vartheta|) \leq 2\rho$.

Quindi la funzione è continua in $(0;0)$.

$$\frac{\partial f}{\partial x}(0;0) = \lim_{h\to0}\left[(0+h)^2\cdot\frac{(0+h)-0}{(0+h)^2+0^2}-0\right]\cdot\frac{1}{h} = \lim_{h\to0} h^2\cdot\frac{h}{h^2}\cdot\frac{1}{h} = 1;$$

$$\frac{\partial f}{\partial y}(0;0) = \lim_{h\to0}\left[0^2\cdot\frac{0-(0+h)}{0^2+(0+h)^2}-0\right]\cdot\frac{1}{h} = \lim_{h\to0}\frac{0}{h} = \lim_{h\to0} 0 = 0.$$

Per vedere se la funzione è differenziabile in $(0;0)$:

$$\lim_{(x;y)\to(0;0)}\left[x^2\cdot\frac{x-y}{x^2+y^2}-0-(1;0)\cdot(x-0;y-0)\right]\cdot\frac{1}{\sqrt{x^2+y^2}} = \lim_{(x;y)\to(0;0)}\frac{\cancel{x^3}-x^2y-\cancel{x^3}-xy^2}{(x^2+y^2)\cdot\sqrt{x^2+y^2}} \Rightarrow$$

$$\Rightarrow \lim_{\rho\to0} -\frac{\rho^3\cos\vartheta\,\text{sen}\vartheta\,(\cos\vartheta+\text{sen}\vartheta)}{\rho^3} = -\cos\vartheta\,\text{sen}\vartheta\,(\cos\vartheta+\text{sen}\vartheta) \neq 0$$

eccetto che per $\vartheta = 0; \frac{\pi}{2}; \pi; \frac{3}{2}\pi; \frac{3}{4}\pi; \frac{7}{4}\pi \Rightarrow f$ non è differenziabile in $(0;0)$.

IM3) $f(x;y) = \log(x^2+y^2) - \text{arctg}\frac{y}{x} = 0$; $f(1;0) = \log 1 + \text{arctg}\,0 = 0$.

$$\nabla f(x;y) = \left(\frac{2x}{x^2+y^2} - \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2};\ \frac{2y}{x^2+y^2} - \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\right) = \left(\frac{2x+y}{x^2+y^2};\ \frac{2y-x}{x^2+y^2}\right).\ \nabla f(1;0) = (2;-1).$$

Essendo $f'_y(1;0)=-1\neq 0$ esiste $y=y(x)$ con $y'(1)=-\frac{2}{-1}=2$.

$$H(f(x;y))=\left\|\begin{matrix}\frac{2(x^2+y^2)-2x(2x+y)}{(x^2+y^2)^2} & \frac{1\cdot(x^2+y^2)-2y(2x+y)}{(x^2+y^2)^2}\\ \frac{-1(x^2+y^2)-2x(2y-x)}{(x^2+y^2)^2} & \frac{2\cdot(x^2+y^2)-2y(2y-x)}{(x^2+y^2)^2}\end{matrix}\right\|=\left\|\begin{matrix}\frac{2y^2-2xy-2x^2}{(x^2+y^2)^2} & \frac{x^2-4xy-y^2}{(x^2+y^2)^2}\\ \frac{x^2-4xy-y^2}{(x^2+y^2)^2} & \frac{2x^2+2xy-2y^2}{(x^2+y^2)^2}\end{matrix}\right\|.$$

$$H(1;0)=\left\|\begin{matrix}-2 & 1\\ 1 & 2\end{matrix}\right\|\Rightarrow y''(1)=-\frac{-2+2\cdot1\cdot2+2\cdot4}{-1}=-\frac{10}{-1}=10.$$

IM4) $f(x;y)=e^{x-y}$: funzione continua e differenziabile due volte.

$\nabla f(x;y)=(e^{x-y};-e^{x-y})$; $\nabla f(0;0)=(1;-1)$. $\mathcal{D}_v f(0;0)=(1;-1)\cdot(\cos\alpha;\operatorname{sen}\alpha)=\cos\alpha-\operatorname{sen}\alpha$.

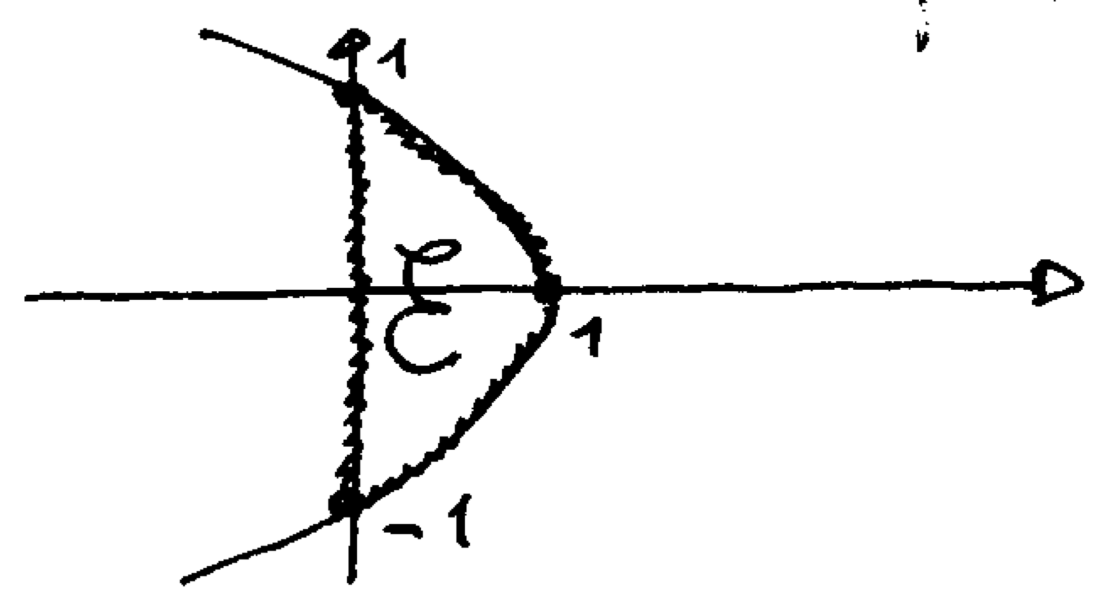
$$H(f(x;y))=\left\|\begin{matrix}e^{x-y} & -e^{x-y}\\ -e^{x-y} & e^{x-y}\end{matrix}\right\|;\quad H(f(0;0))=\left\|\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right\|.$$

$$\mathcal{D}^2_{v,-v}f(0;0)=\left\|\cos\alpha\ \operatorname{sen}\alpha\right\|\cdot\left\|\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right\|\cdot\left\|\begin{matrix}-\cos\alpha\\ -\operatorname{sen}\alpha\end{matrix}\right\|=\left\|\cos\alpha\ \operatorname{sen}\alpha\right\|\cdot\left\|\begin{matrix}\operatorname{sen}\alpha-\cos\alpha\\ \cos\alpha-\operatorname{sen}\alpha\end{matrix}\right\|=$$

$$=\cos\alpha\operatorname{sen}\alpha-\cos^2\alpha+\operatorname{sen}\alpha\cos\alpha-\operatorname{sen}^2\alpha=2\operatorname{sen}\alpha\cos\alpha-1=\operatorname{sen}2\alpha-1.$$

$\mathcal{D}_v f(0;0)=\mathcal{D}^2_{v,-v}f(0;0)\Rightarrow\cos\alpha-\operatorname{sen}\alpha=\operatorname{sen}2\alpha-1\Rightarrow$ vero per $\alpha=\frac{\pi}{4};\frac{\pi}{2};\pi;\frac{5}{4}\pi$.

IIM1) $\begin{cases}\text{Max/min } f(x;y)=xy-y^2\\ \text{s.v.}\begin{cases}x\leq 1-y^2\\ 0\leq x\end{cases}\Rightarrow\begin{cases}x+y^2-1\leq 0\\ -x\leq 0\end{cases}\end{cases}$



E insieme limitato e chiuso; $f(x;y)$ funzione continua, vincoli qualificati.

$$\Lambda=xy-y^2-\lambda_1(x+y^2-1)-\lambda_2\cdot(-x).$$

<u>Caso $\lambda_1=\lambda_2=0$</u>

$$\begin{cases}\Lambda'_x=y=0\\ \Lambda'_y=x-2y=0\\ x\leq 1-y^2\\ x\geq 0\end{cases}\Rightarrow\begin{cases}x=0\\ y=0\\ 0\leq 1\\ 0\geq 0\end{cases}.\quad H=\left\|\begin{matrix}0 & 1\\ 1 & -2\end{matrix}\right\|;\ |H|=-1<0:\ \text{Punto di sella}.$$

AM3

Caso $\lambda_1 \neq 0;\ \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = y - \lambda_1 = 0 \\ \Lambda'_y = x - 2y - 2\lambda_1 y = 0 \\ x = 1 - y^2 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = y \\ x = 2y + 2y^2 \\ 2y + 2y^2 = 1 - y^2 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = y \\ x = 2y + 2y^2 \\ 3y^2 + 2y - 1 = 0 \\ x \geq 0 \end{cases} ;\ 3y^2 + 2y - 1 = 0 \Rightarrow$$

$$\Rightarrow y = \frac{-1 \pm \sqrt{1+3}}{3} = \frac{-1 \pm 2}{3} = \begin{cases} -1 \\ \frac{1}{3} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \\ \lambda_1 = -1 \\ 0 \geq 0 \end{cases} \text{Min?} \cup \begin{cases} x = \frac{8}{9} \\ y = \frac{1}{3} \\ \lambda_1 = \frac{1}{3} \\ \frac{8}{9} \geq 0 \end{cases} \text{Max?}$$

Caso $\lambda_1 = 0;\ \lambda_2 \neq 0$

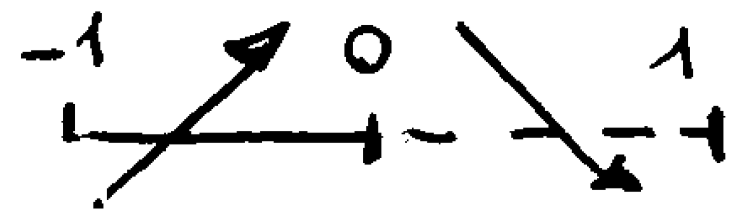
$$\begin{cases} \Lambda'_x = y + \lambda_2 = 0 \\ \Lambda'_y = x - 2y = 0 \\ x = 0 \\ x \leq 1 - y^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_2 = 0 \\ 0 \leq 1 \end{cases} \text{ già visto.}$$

Caso $\lambda_1 \neq 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = y - \lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = x - 2y - 2\lambda_1 y = 0 \\ \begin{cases} x = 1 - y^2 \\ x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \cup \begin{cases} x = 0 \\ y = -1 \end{cases} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ 1 - \lambda_1 + \lambda_2 = 0 \\ 0 - 2 - 2\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases} \text{Min?} \cup \begin{cases} x = 0 \\ y = -1 \\ -1 - \lambda_1 + \lambda_2 = 0 \\ 2 + 2\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \\ \lambda_1 = -1 \\ \lambda_2 = 0 \end{cases} \text{Min?}$$

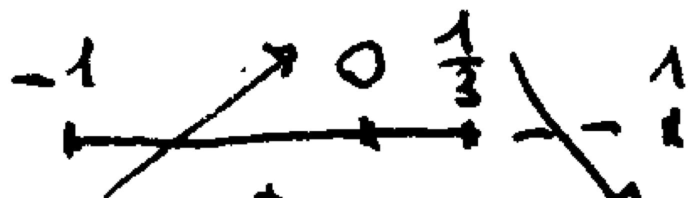
già visto

Se $x = 0$: $f(y) = -y^2$ che ha un massimo per $y = 0$



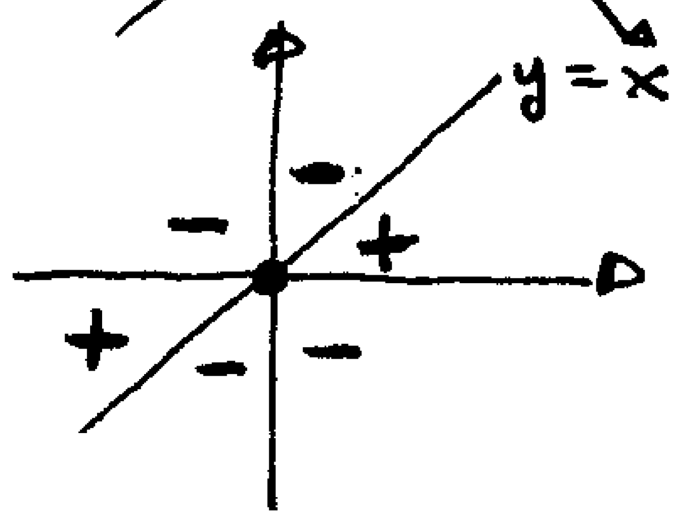
Se $x = 1 - y^2$: $f(y) = (1-y^2) \cdot y - y^2 = y - y^3 - y^2$. $f'(y) = 1 - 2y - 3y^2 \geq 0$ per $3y^2 + 2y - 1 \leq 0$

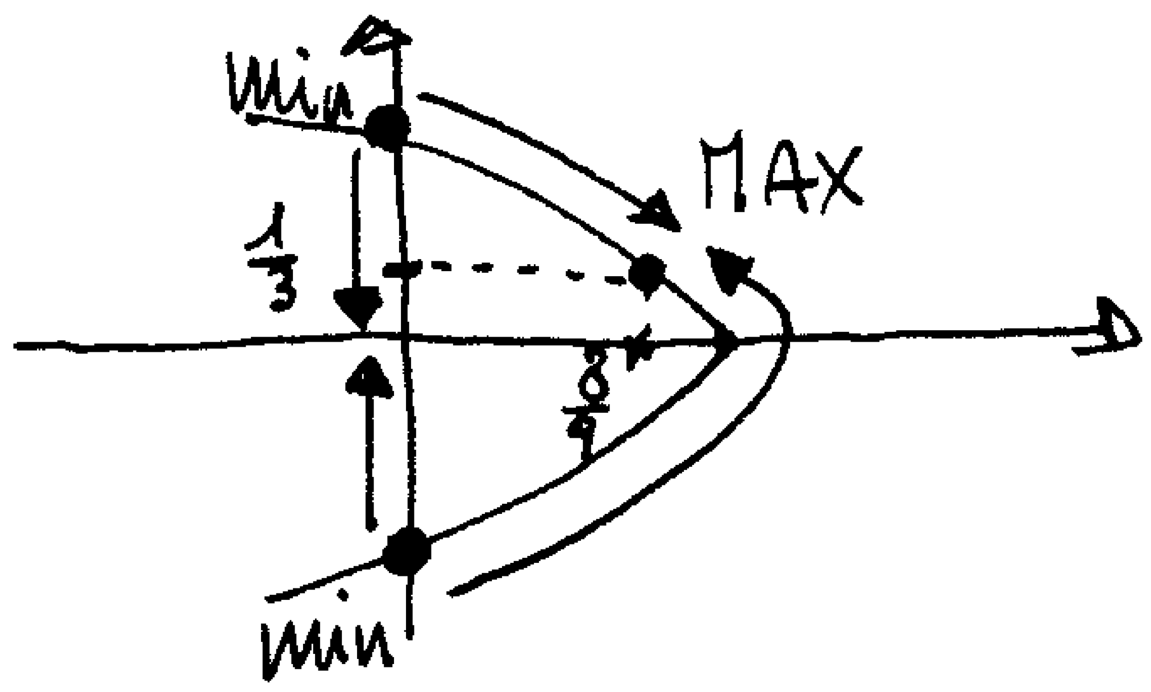
vera per $-1 \leq y \leq \frac{1}{3}$;



$f(x;y) = y(x - y) \geq 0$

$f(0;0) = 0$.





Il punto $(0;0)$ non è né di Massimo né di minimo. $\left(\frac{8}{9}; \frac{1}{3}\right)$ è punto di Massimo,

$(0;1)$ e $(0;-1)$ sono punti di Minimo.

II n 2) $\begin{cases} y''' - 5y'' - y' + 5y = 0 \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 24 \end{cases} \Rightarrow \lambda^3 - 5\lambda^2 - \lambda + 5 = 0$

per $\lambda = 1: 1 - 5 - 1 + 5 = 0$

$$\begin{array}{c|ccc|c} & 1 & -5 & -1 & 5 \\ 1 & & 1 & -4 & -5 \\ \hline & 1 & -4 & -5 & 0 \end{array}$$

$\lambda^3 - 5\lambda^2 - \lambda + 5 = (\lambda - 1)(\lambda^2 - 4\lambda - 5) = 0$; $\lambda^2 = 2 \pm \sqrt{4+5} = 2 \pm 3 \Rightarrow \lambda = 5$ e $\lambda = -1$.

Soluzioni: $\lambda = 1$; $\lambda = -1$; $\lambda = 5$. Soluzione generale: $y = c_1 e^x + c_2 e^{-x} + c_3 e^{5x}$.

$y' = c_1 e^x - c_2 e^{-x} + 5c_3 e^{5x}$; $y'' = c_1 e^x + c_2 e^{-x} + 25 c_3 e^{5x}$.

$\begin{cases} y(0) = c_1 + c_2 + c_3 = 0 \\ y'(0) = c_1 - c_2 + 5c_3 = 0 \\ y''(0) = c_1 + c_2 + 25c_3 = 24 \end{cases}$. $\Rightarrow \begin{cases} \text{III} - \text{I}: 24c_3 = 24 \Rightarrow c_3 = 1 \\ c_1 = -c_2 - 1 \\ -c_2 - 1 - c_2 + 5 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = -3 \\ c_2 = 2 \\ c_3 = 1 \end{cases}$.

Soluzione del problema: $y = -3e^x + 2e^{-x} + e^{5x}$.

II n 3) $\begin{cases} x' = x - 2y + t \\ y' = x - y - t \end{cases} \Rightarrow \begin{cases} x' - x + 2y = t \\ -x + y' + y = -t \end{cases} \Rightarrow \left\| \begin{matrix} D-1 & 2 \\ -1 & D+1 \end{matrix} \right\| \cdot \left\| \begin{matrix} x \\ y \end{matrix} \right\| = \left\| \begin{matrix} t \\ -t \end{matrix} \right\| \Rightarrow$

$\Rightarrow \begin{vmatrix} D-1 & 2 \\ -1 & D+1 \end{vmatrix} (x) = \begin{vmatrix} t & 2 \\ -t & D+1 \end{vmatrix} \Rightarrow (D^2+1)(x) = 1 + t + 2t \Rightarrow x'' + x = 3t + 1$.

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow$ Soluzione dell'omogenea: $x(t) = c_1 \cos t + c_2 \operatorname{sen} t$.

Soluzione particolare: $x_0(t) = at + b$; $x_0'(t) = a$; $x_0''(t) = 0 \Rightarrow$

$\Rightarrow 0 + at + b = 3t + 1 \Rightarrow a = 3$ e $b = 1$. Soluzione della non omogenea:

$x(t) = c_1 \cos t + c_2 \operatorname{sen} t + 3t + 1$. Da $x' = x - 2y + t \Rightarrow y = \frac{1}{2}(x - x' + t) \Rightarrow$

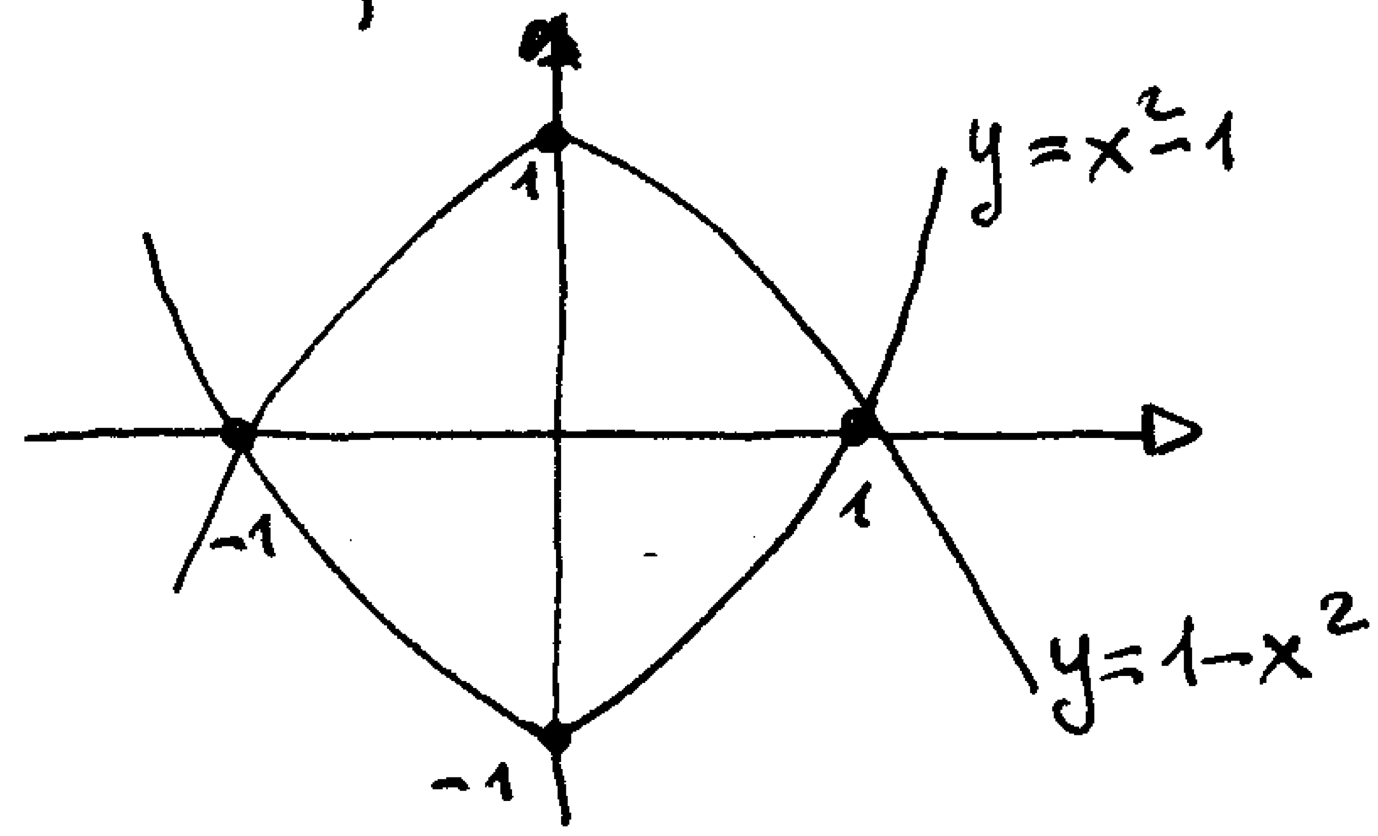
$\Rightarrow y(t) = \frac{1}{2}(c_1 \cos t + c_2 \operatorname{sen} t + 3t + 1 + c_1 \operatorname{sen} t - c_2 \cos t - 3 + t) \Rightarrow$

$\Rightarrow y(t) = \frac{1}{2} c_1 (\cos t + \operatorname{sen} t) + \frac{1}{2} c_2 (\operatorname{sen} t - \cos t) + 2t - 1$.

$\begin{matrix} x(0) = c_1 + 1 = 1 \\ y(0) = \frac{1}{2}c_1 - \frac{1}{2}c_2 - 1 = -1 \end{matrix} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} \Rightarrow \begin{cases} x(t) = 3t + 1 \\ y(t) = 2t - 1 \end{cases}$.

$\boxed{AM5}$

II M 4) $\iint_D x^2+y^2\,dx\,dy$; $D=\{(x;y): x^2-1 \le y \le 1-x^2\}$.



$$\iint_D x^2+y^2\,dx\,dy = \int_{-1}^{+1}\left(\int_{x^2-1}^{1-x^2} x^2+y^2\,dy\right)dx =$$

$$=\int_{-1}^{+1}\left(x^2y+\frac{1}{3}y^3\right|_{x^2-1}^{1-x^2} dx =$$

$$=\int_{-1}^{+1} x^2\cdot(1-x^2)+\frac{1}{3}(1-x^2)^3-x^2(x^2-1)-\frac{1}{3}(x^2-1)^3\,dx =$$

$$=\int_{-1}^{+1} x^2-x^4-x^4+x^2+\frac{1}{3}\left(1-3x^2+3x^4-x^6-x^6+3x^4-3x^2+1\right)dx=$$

$$=\int_{-1}^{+1} \cancel{2x^2}-\cancel{2x^4}+\frac{2}{3}-\cancel{2x^2}+\cancel{2x^4}-\frac{2}{3}x^6\,dx = \frac{2}{3}\int_{-1}^{+1}1-x^6\,dx =$$

$$= \frac{2}{3}\left(x-\frac{1}{7}x^7\right|_{-1}^{+1}=\frac{2}{3}\left(1-\frac{1}{7}-\left(-1+\frac{1}{7}\right)\right)=\frac{2}{3}\cdot\left(2-\frac{2}{7}\right)=\frac{2}{3}\cdot\frac{12}{7}=\frac{8}{7}.$$