Compito di Analisi Matematica del 23/2/2016 $\boxed{AM1}$

IM1) $z = i^{15}\cdot\frac{(1+i)^5}{(1-i)^6}$. $i = 1\cdot\left(\cos\frac{\pi}{2} + i\,\text{sen}\,\frac{\pi}{2}\right)$; $i^{15} = \cos\frac{15}{2}\pi + i\,\text{sen}\,\frac{15}{2}\pi = \cos\frac{3}{2}\pi + i\,\text{sen}\,\frac{3}{2}\pi$.

$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\text{sen}\,\frac{\pi}{4}\right)$; $(1+i)^5 = (\sqrt{2})^5\cdot\left(\cos\frac{5}{4}\pi + i\,\text{sen}\,\frac{5}{4}\pi\right)$.

$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{7}{4}\pi + i\,\text{sen}\,\frac{7}{4}\pi\right)$; $(1-i)^6 = (\sqrt{2})^6\cdot\left(\cos\frac{42}{4}\pi + i\,\text{sen}\,\frac{42}{4}\pi\right) = (\sqrt{2})^6\cdot\left(\cos\frac{\pi}{2} + i\,\text{sen}\,\frac{\pi}{2}\right)$.

$i^{15}\cdot\frac{(1+i)^5}{(1-i)^5} = \frac{(\sqrt{2})^5}{(\sqrt{2})^6}\cdot\left(\cos\left(\frac{3}{2}\pi + \frac{5}{4}\pi - \frac{\pi}{2}\right) + i\,\text{sen}\left(\frac{3}{2}\pi + \frac{5}{4}\pi - \frac{\pi}{2}\right)\right) = \frac{1}{\sqrt{2}}\cdot\left(\cos\frac{9}{4}\pi + i\,\text{sen}\,\frac{9}{4}\pi\right) =$

$= \frac{1}{\sqrt{2}}\cdot\left(\cos\frac{\pi}{4} + i\,\text{sen}\,\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\cdot\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + i\frac{1}{2}$.

IM2) $f(x;y) = \begin{cases} \frac{xy(x^3-y^3)}{(x^2+y^2)^2} & : (x;y) \neq (0;0) \\ 0 & : (x;y) = (0;0) \end{cases}$. Vediamo se $f(x;y) \in C((0;0))$.

$$\lim_{(x;y)\to(0;0)} \frac{xy(x^3-y^3)}{(x^2+y^2)^2} \Rightarrow \lim_{\rho\to 0} \frac{\rho^2\cos\vartheta\,\text{sen}\,\vartheta\cdot(\rho^3(\cos^3\vartheta - \text{sen}^3\vartheta))}{(\rho^2)^2} =$$

$= \lim_{\rho\to 0} \rho\cdot\cos\vartheta\,\text{sen}\,\vartheta\cdot(\cos^3\vartheta - \text{sen}^3\vartheta) = 0$. La convergenza è uniforme in quanto: $\left|\rho\cdot\cos\vartheta\,\text{sen}\,\vartheta\,(\cos^3\vartheta - \text{sen}^3\vartheta)\right| \leq \rho\cdot 1\cdot 1\cdot 2 = 2\rho < \varepsilon$ per $\rho < \delta(\varepsilon) = \frac{\varepsilon}{2}$.

$$\frac{\partial f}{\partial x}(0;0) = \lim_{h\to 0}\left[\frac{(0+h)\cdot 0\cdot((0+h)^3-0)}{((0+h)^2+0)^2} - 0\right]\cdot\frac{1}{h} = \lim_{h\to 0} 0 = 0;$$

$$\frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0}\left[\frac{0\cdot(0+h)\cdot(0-(0+h)^3)}{(0+(0+h)^2)^2} - 0\right]\cdot\frac{1}{h} = \lim_{h\to 0} 0 = 0.$$

$$\lim_{(x;y)\to(0;0)} \frac{f(x;y) - f(0;0) - \nabla f(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}} = \lim_{(x;y)\to(0;0)} \frac{xy(x^3-y^3)}{(x^2+y^2)^2\cdot\sqrt{x^2+y^2}} \Rightarrow$$

$$\Rightarrow \lim_{\rho\to 0} \frac{\rho^2\cos\vartheta\,\text{sen}\,\vartheta\cdot(\rho^3(\cos^3\vartheta - \text{sen}^3\vartheta))}{(\rho^2)^2\cdot\rho} = \lim_{\rho\to 0} \cos\vartheta\,\text{sen}\,\vartheta(\cos^3\vartheta - \text{sen}^3\vartheta) =$$

$= \cos\vartheta\cdot\text{sen}\,\vartheta\cdot(\cos^3\vartheta - \text{sen}^3\vartheta) = 0$ solo per $\vartheta = 0; \frac{\pi}{2}; \frac{\pi}{4}; \pi; \frac{5}{4}\pi; \frac{3}{2}\pi$. Quindi la funzione non è differenziabile in $(0;0)$.

IM3) $\begin{cases} f(x;y;z) = e^{x^4-y^4} + e^{z^4-y^4} - 2xyz = 0 \\ g(x;y;z) = x^3 - y^3 + z^3 + 3xz = 0 \end{cases}$; $\begin{cases} f(-1;1;-1) = 1 + 1 - 2 = 0 \\ g(-1;1;-1) = -1 - 1 - 1 + 3 = 0 \end{cases}$.

AM2

$$\frac{\partial(f;g)}{\partial(x;y;z)}=\left\|\begin{matrix}4x^3e^{x^4-y^4}-2yz & -4y^3e^{x^4-y^4}-4y^3e^{z^4-y^4}-2xz & 4z^3e^{z^4-y^4}-2xy\\ 3x^2+3z & -3y^2 & 3z^2+3x\end{matrix}\right\|$$

$$\frac{\partial(f;g)}{\partial(x;y;z)}(-1;1;-1)=\left\|\begin{matrix}-4+2 & -4-4-2 & -4+2\\ 3-3 & -3 & 3-3\end{matrix}\right\|=\left\|\begin{matrix}-2 & -10 & -2\\ 0 & -3 & 0\end{matrix}\right\|.$$

Essendo $\begin{vmatrix}-10 & -2\\ -3 & 0\end{vmatrix}=0-6\neq 0$ si può definire $F:\mathbb{R}\to\mathbb{R}^2;\ x\to(y(x);z(x))$.

$$\frac{dy}{dx}=-\frac{\begin{vmatrix}-2 & -2\\ 0 & 0\end{vmatrix}}{-6}=-\frac{0}{-6}=0\ ;\quad \frac{dz}{dx}=-\frac{\begin{vmatrix}-10 & -2\\ -3 & 0\end{vmatrix}}{-6}=-\frac{-6}{-6}=-1.$$

I M4) $f(x;y)=e^{\alpha(x-y)}$: funzione differenziabile due volte $\forall(x;y)\in\mathbb{R}^2$.

$\nabla f(x;y)=(\alpha e^{\alpha(x-y)};-\alpha e^{\alpha(x-y)})$; $\nabla f(0;0)=(\alpha;-\alpha)$. $(1;-1)\to(\frac{1}{\sqrt2};-\frac{1}{\sqrt2})=v$.

$$D_v f(0;0)=\nabla f(0;0)\cdot v=(\alpha;-\alpha)\cdot\left(\frac{1}{\sqrt2};-\frac{1}{\sqrt2}\right)=\frac{1}{\sqrt2}\alpha+\frac{1}{\sqrt2}\alpha=\frac{2}{\sqrt2}\alpha=\sqrt2\cdot\alpha.$$

$$H(x;y)=\left\|\begin{matrix}\alpha^2e^{\alpha(x-y)} & -\alpha^2e^{\alpha(x-y)}\\ -\alpha^2e^{\alpha(x-y)} & \alpha^2e^{\alpha(x-y)}\end{matrix}\right\|;\ H(0;0)=\left\|\begin{matrix}\alpha^2 & -\alpha^2\\ -\alpha^2 & \alpha^2\end{matrix}\right\|.$$

$$D^2_{v,-v}f(0;0)=\left\|\begin{matrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\end{matrix}\right\|\cdot\left\|\begin{matrix}\alpha^2 & -\alpha^2\\ -\alpha^2 & \alpha^2\end{matrix}\right\|\cdot\left\|\begin{matrix}-\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{matrix}\right\|=\left\|\begin{matrix}\frac{1}{\sqrt2} & -\frac{1}{\sqrt2}\end{matrix}\right\|\cdot\left\|\begin{matrix}-\frac{1}{\sqrt2}\alpha^2-\frac{1}{\sqrt2}\alpha^2\\ \frac{1}{\sqrt2}\alpha^2+\frac{1}{\sqrt2}\alpha^2\end{matrix}\right\|=$$

$$=\frac{1}{\sqrt2}\cdot\left(-\frac{2}{\sqrt2}\alpha^2\right)+\left(-\frac{1}{\sqrt2}\right)\cdot\left(\frac{2}{\sqrt2}\alpha^2\right)=-\alpha^2-\alpha^2=-2\alpha^2.$$

$$D_v f(0;0)+D^2_{v,-v}f(0;0)=\sqrt2\cdot\alpha-2\alpha^2=\frac14\Rightarrow 8\alpha^2-4\sqrt2\alpha+1=0\Rightarrow\alpha=\frac{2\sqrt2\pm\sqrt{8-8}}{8}=\frac{\sqrt2}{4}.$$

II M1) $\begin{cases}\text{Max/min } f(x;y;z)=x+y+z\\ \text{s.v.}: x^2+2y^2+3z^2=\frac{11}{6}\end{cases}$ • L'insieme è un insieme limitato e chiuso. $f(x;y;z)$ è funzione continua $\Rightarrow$ esistono Max e min.

$$\Lambda=x+y+z-\lambda\left(x^2+2y^2+3z^2-\frac{11}{6}\right).$$

$$\begin{cases}\Lambda'_x=1-2\lambda x=0\\ \Lambda'_y=1-4\lambda y=0\\ \Lambda'_z=1-6\lambda z=0\\ x^2+2y^2+3z^2=\frac{11}{6}\end{cases}\Rightarrow\begin{cases}x=\frac{1}{2\lambda}\\ y=\frac{1}{4\lambda}.\\ z=\frac{1}{6\lambda}\\ \frac{1}{\lambda^2}\left(\frac14+2\cdot\frac{1}{16}+3\cdot\frac{1}{36}\right)=\frac{1}{\lambda^2}\cdot\left(\frac{6+3+2}{24}\right)=\frac{1}{\lambda^2}\cdot\frac{11}{24}=\frac{11}{6}\Rightarrow\lambda^2=\frac{11}{24}\cdot\frac{6}{11}=\frac14.\end{cases}$$

$\Rightarrow\lambda^2=\frac14\Rightarrow\lambda=\pm\frac12$. $P_1=\left(1;\frac12;\frac13;\frac12\right)$ e $P_2=\left(-1;-\frac12;-\frac13;-\frac12\right)$.

AM3

Valendo il Teorema di Weierstrass, essendo $f(1;\frac{1}{2};\frac{1}{3})=\frac{11}{6}$ e $f(-1;-\frac{1}{2};-\frac{1}{3})=-\frac{11}{6}$ ne consegue che P_1 è il Punto di Massimo e P_2 è punto di minimo.

Se si fosse voluto usare la matrice Hessiana orlata (in questo caso inutile in quanto ci sono due soluzioni sole), si ha:

$$\bar{H}=\left\|\begin{matrix}0 & 2x & 4y & 6z\\ 2x & -2\lambda & 0 & 0\\ 4y & 0 & -4\lambda & 0\\ 6z & 0 & 0 & -6\lambda\end{matrix}\right\| \cdot \bar{H}^1\left(1;\frac{1}{2};\frac{1}{3};\frac{1}{2}\right)=\left\|\begin{matrix}0 & 2 & 2 & 2\\ 2 & -1 & 0 & 0\\ 2 & 0 & -2 & 0\\ 2 & 0 & 0 & -3\end{matrix}\right\|;\ \bar{H}^2\left(-1;-\frac{1}{2};-\frac{1}{3};-\frac{1}{2}\right)=\left\|\begin{matrix}0 & -2 & -2 & -2\\ -2 & 1 & 0 & 0\\ -2 & 0 & 2 & 0\\ -2 & 0 & 0 & 3\end{matrix}\right\|.$$

$$|\bar{H}_3^1|=\begin{vmatrix}0 & 2 & 2\\ 2 & -1 & 0\\ 2 & 0 & -2\end{vmatrix}=\begin{vmatrix}0 & 2 & 2\\ 0 & -1 & 2\\ 2 & 0 & -2\end{vmatrix}=2\cdot\begin{vmatrix}2 & 2\\ -1 & 2\end{vmatrix}=2(4+2)=12>0$$

$$|\bar{H}_4^1|=\begin{vmatrix}0 & 2 & 2 & 2\\ 2 & -1 & 0 & 0\\ 2 & 0 & -2 & 0\\ 2 & 0 & 0 & -3\end{vmatrix}=\begin{vmatrix}0 & 2 & 2 & 2\\ 0 & -1 & 2 & 0\\ 2 & 0 & -2 & 0\\ 0 & 0 & 2 & -3\end{vmatrix}=2\cdot\begin{vmatrix}2 & 2 & 2\\ -1 & 2 & 0\\ 0 & 2 & -3\end{vmatrix}=2\cdot\begin{vmatrix}0 & 6 & 2\\ -1 & 2 & 0\\ 0 & 2 & -3\end{vmatrix}=2\cdot1\cdot(-18-4)=-44<0.$$

Quindi $|\bar{H}_3^1|=12>0\ (+)$; $|\bar{H}_4^1|=-44<0\ (-)$: $(+;-)\Rightarrow$ Punto di Massimo.

Poi $|\bar{H}_3^2|=(-1)^3\cdot|\bar{H}_3^1|=-12<0$; $|\bar{H}_4^2|=(-1)^4\cdot|\bar{H}_4^1|=-44<0$: $(-;-)\Rightarrow$ Punto di minimo.

AM2) $\begin{cases} y'-tg\,x\cdot y=\operatorname{sen}x\\ y(0)=0\end{cases}$: Equazione lineare del I ordine.

$$y(x)=e^{-\int -tg\,x\,dx}\cdot\left(\int \operatorname{sen}x\cdot e^{\int -tg\,x\,dx}dx+k\right)=e^{-\int\frac{-\operatorname{sen}x}{\cos x}dx}\cdot\left(\int \operatorname{sen}x\cdot e^{\int\frac{-\operatorname{sen}x}{\cos x}dx}dx+k\right)=$$

$$=e^{-\log|\cos x|}\cdot\left(\int \operatorname{sen}x\cdot e^{\log|\cos x|}dx+k\right)=\frac{1}{\cos x}\cdot\left(\int \operatorname{sen}x\cdot\cos x\,dx+k\right)=$$

$$=\frac{1}{\cos x}\cdot\left(\int -\cos x\,d(\cos x)+k\right)=\frac{1}{\cos x}\cdot\left(-\frac{1}{2}\cos^2 x+k\right)=-\frac{1}{2}\cos x+\frac{k}{\cos x}.$$

$$y(0)=0\Rightarrow 0=-\frac{1}{2}\cdot1+\frac{k}{1}\Rightarrow k=\frac{1}{2}:\ y_0(x)=-\frac{1}{2}\cos x+\frac{1}{2\cos x}.$$

$\boxed{AM4}$

IIM3) $\begin{cases} x' = 2x+y+t \\ y' = -2x-y-1 \end{cases} \Rightarrow \begin{cases} x'-2x-y = t \\ 2x+y'+y = -1 \end{cases} \Rightarrow \left\| \begin{matrix} D-2 & -1 \\ 2 & D+1 \end{matrix} \right\| \cdot \left\| \begin{matrix} x \\ y \end{matrix} \right\| = \left\| \begin{matrix} t \\ -1 \end{matrix} \right\|.$

$\left| \begin{matrix} D-2 & -1 \\ 2 & D+1 \end{matrix} \right| (x) = \left| \begin{matrix} t & -1 \\ -1 & D+1 \end{matrix} \right| \Rightarrow (D^2-D-2+2)(x) = (D+1)(t)-1 \Rightarrow (D^2-D)(x) = D(D-1)x = 1+t-1 = t.$

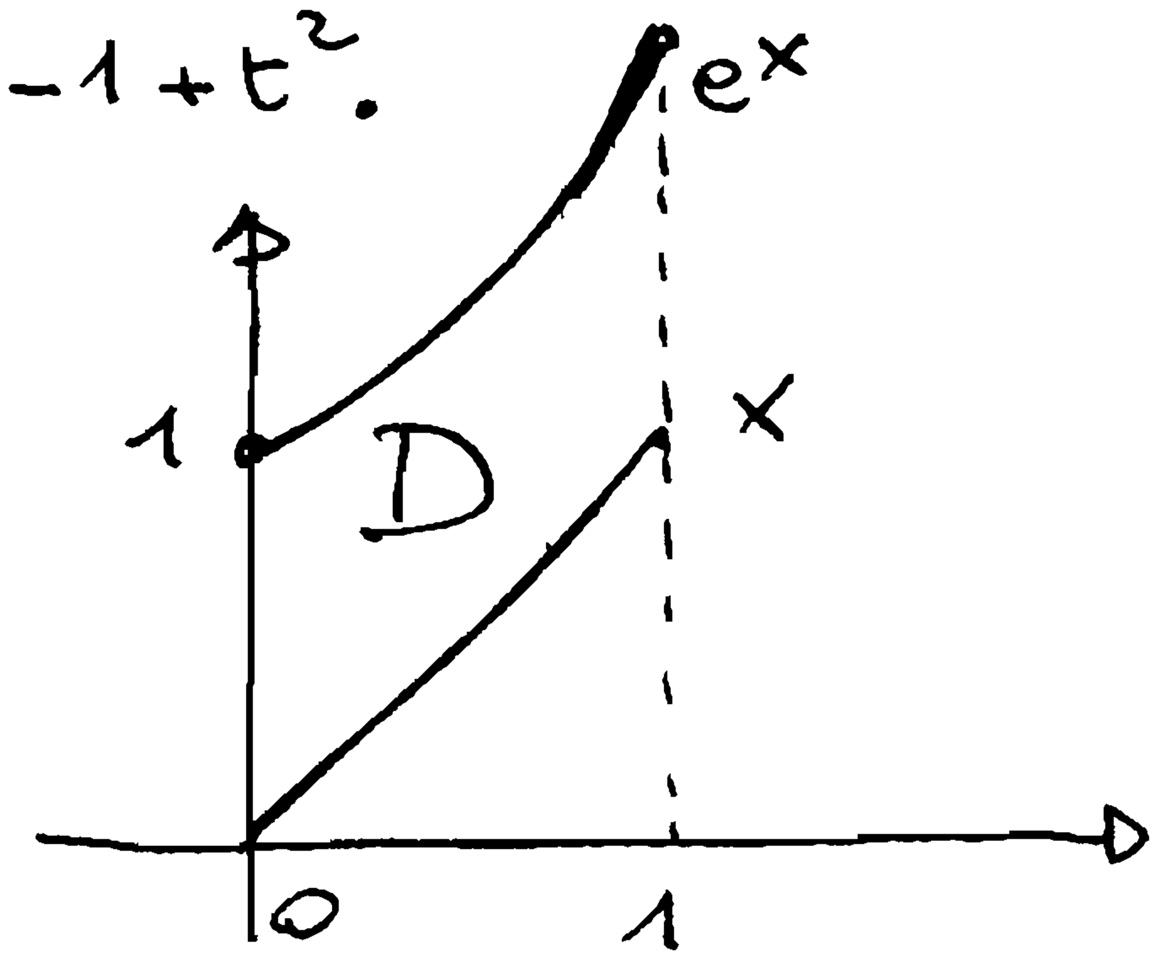
Soluzione della omogenea: $x(t) = c_1 e^{0t} + c_2 e^{1\cdot t} = c_1 + c_2 e^t$. Dato che t è annichilato da D^2, avremo $D^2 \cdot D \cdot (D-1)(x) = D^2(t) = 0$. La soluzione per D^3 sarà data da $x_0 = a + bt + ct^2 \Rightarrow x_0' = b+2ct$; $x_0'' = 2c$. Da $x''-x' = t$ si ha:

$\Rightarrow 2c - b - 2ct = t \Rightarrow \begin{cases} -2ct = t \\ 2c-b = 0 \end{cases} \Rightarrow \begin{cases} c = -\frac{1}{2} \\ b = 2c = -1 \end{cases} \Rightarrow x_0 = a - t - \frac{1}{2}t^2.$

Quindi $x(t) = c_1 + c_2 e^t - t - \frac{1}{2}t^2$. Da $y = x' - 2x - t$ si ottiene:

$y = c_2 e^t - 1 - t - 2c_1 - 2c_2 e^t + 2t + t^2 - t \Rightarrow y(t) = -2c_1 - c_2 e^t - 1 + t^2.$

IIM4) $\iint_D x+2y \, dx\,dy$. $D = \{(x;y) : 0 \leq x \leq 1 \,;\, x \leq y \leq e^x\}$



$\iint_D x+2y \, dx\,dy = \int_0^1 \int_x^{e^x} x+2y \, dy \, dx =$

$= \int_0^1 \left(xy + y^2 \right|_x^{e^x} dx = \int_0^1 (xe^x + e^{2x}) - (x^2+x^2) \, dx =$

$= \int_0^1 xe^x + e^{2x} - 2x^2 \, dx = \left(xe^x - e^x + \frac{1}{2}e^{2x} - \frac{2}{3}x^3 \right|_0^1 = \left(e - e + \frac{1}{2}e^2 - \frac{2}{3}\right) - \left(0 - 1 + \frac{1}{2} - 0\right) =$

$= \frac{1}{2}e^2 - \frac{2}{3} + \frac{1}{2} = \frac{1}{2}e^2 - \frac{1}{6}.$