Compito di Analisi Matematica del 19/3/2016 $\boxed{\text{CAM1}}$

IM1) $z=i^{19}-3i^{6}-i^{18}+3i^{3}=i^{4\cdot4+3}-3i^{4+2}-i^{4\cdot4+2}+3i^{3}=$

$=1^{4}\cdot i^{3}-3\cdot1\cdot i^{2}-1^{4}\cdot i^{2}+3i^{3}=-i+3+1-3i=4-4i=4\sqrt{2}\cdot\left(\cos\frac{7}{4}\pi+i\,\mathrm{sen}\,\frac{7}{4}\pi\right).$

$\sqrt[4]{z}=\sqrt[8]{2^5}\cdot\left(\cos\left(\frac{7}{16}\pi+K\cdot\frac{\pi}{2}\right)+i\,\mathrm{sen}\left(\frac{7}{16}\pi+K\frac{\pi}{2}\right)\right);\ 0\leq K\leq 3.$

Gli argomenti delle 4 radici saranno: $\frac{7}{16}\pi$; $\frac{15}{16}\pi$; $\frac{23}{16}\pi$ e $\frac{31}{16}\pi$. Per avere parte immaginaria positiva deve essere $\mathrm{sen}\,\alpha>0 \Rightarrow \alpha=\frac{7}{16}\pi$ e $\alpha=\frac{15}{16}\pi$.

IM2) $f(x;y)=e^{x^2-y^2}$. Funzione differenziabile due volte $\forall\,(x;y)\in\mathbb{R}^2$.

$\nabla f(x;y)=\left(2x\,e^{x^2-y^2};-2y\,e^{x^2-y^2}\right)$. $H(x;y)=\begin{Vmatrix}(2+4x^2)e^{x^2-y^2} & -4xy\,e^{x^2-y^2}\\ -4xy\,e^{x^2-y^2} & (4y^2-2)e^{x^2-y^2}\end{Vmatrix}$.

Posto $e^{x^2-y^2}=K$ e $v=\left(\frac{1}{\sqrt2};\frac{1}{\sqrt2}\right)$ si ha:

$\mathcal{D}^2_{v,v}f(x;y)+\mathcal{D}^2_{-v,-v}f(x;y)=\begin{Vmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{Vmatrix}\cdot\begin{Vmatrix}(2+4x^2)\cdot K & -4xy\cdot K\\ -4xy\cdot K & (4y^2-2)\cdot K\end{Vmatrix}\cdot\begin{Vmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{Vmatrix}\cdot 2$ dato che

$\mathcal{D}^2_{-v,-v}f(x;y)=(-1)\cdot v\cdot H\cdot(-1)\cdot v^T=v\cdot H\cdot v^T=\mathcal{D}^2_{v,v}f(x;y)$. Quindi

$2\mathcal{D}^2_{v,v}f(x;y)=\frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2}\cdot K\cdot 2\cdot\left(\begin{Vmatrix}1 & 1\end{Vmatrix}\cdot\begin{Vmatrix}2+4x^2 & -4xy\\ -4xy & 4y^2-2\end{Vmatrix}\cdot\begin{Vmatrix}1\\ 1\end{Vmatrix}\right)=K\cdot\left(4x^2+2-8xy+4y^2-2\right)=$

$=K\cdot4\cdot(x^2+y^2-2xy)=4\cdot K\cdot(x-y)^2=0$ sse $y=x$ dato che $K\neq0\ \forall(x;y)$.

IM3) $f(x;y)=\log(x^2+y^2)-xy=0$. $P=(0;1)$. $f(0;1)=\log1+0=0$.

$\nabla f(x;y)=\left(\frac{2x}{x^2+y^2}-y;\frac{2y}{x^2+y^2}-x\right)$; $\nabla f(0;1)=(0-1;2-0)=(-1;2)$.

Quindi $y'(0)=-\frac{f'_x(0;1)}{f'_y(0;1)}=-\frac{-1}{2}=\frac{1}{2}$.

$H(x;y)=\begin{Vmatrix}\frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2}-0 & -\frac{4xy}{(x^2+y^2)^2}-1\\ -\frac{4xy}{(x^2+y^2)^2}-1 & \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}-0\end{Vmatrix}$; $H(0;1)=\begin{Vmatrix}2 & -1\\ -1 & -2\end{Vmatrix}$.

$y''(0)=-\frac{f''_{xx}+2f''_{xy}\cdot y'+f''_{yy}\cdot(y')^2}{f'_y}=-\frac{2+2\cdot(-1)\cdot\frac{1}{2}+(-2)\cdot\left(\frac{1}{2}\right)^2}{2}=-\frac{2-1-\frac{1}{2}}{2}=-\frac{1}{4}.$

CAM2

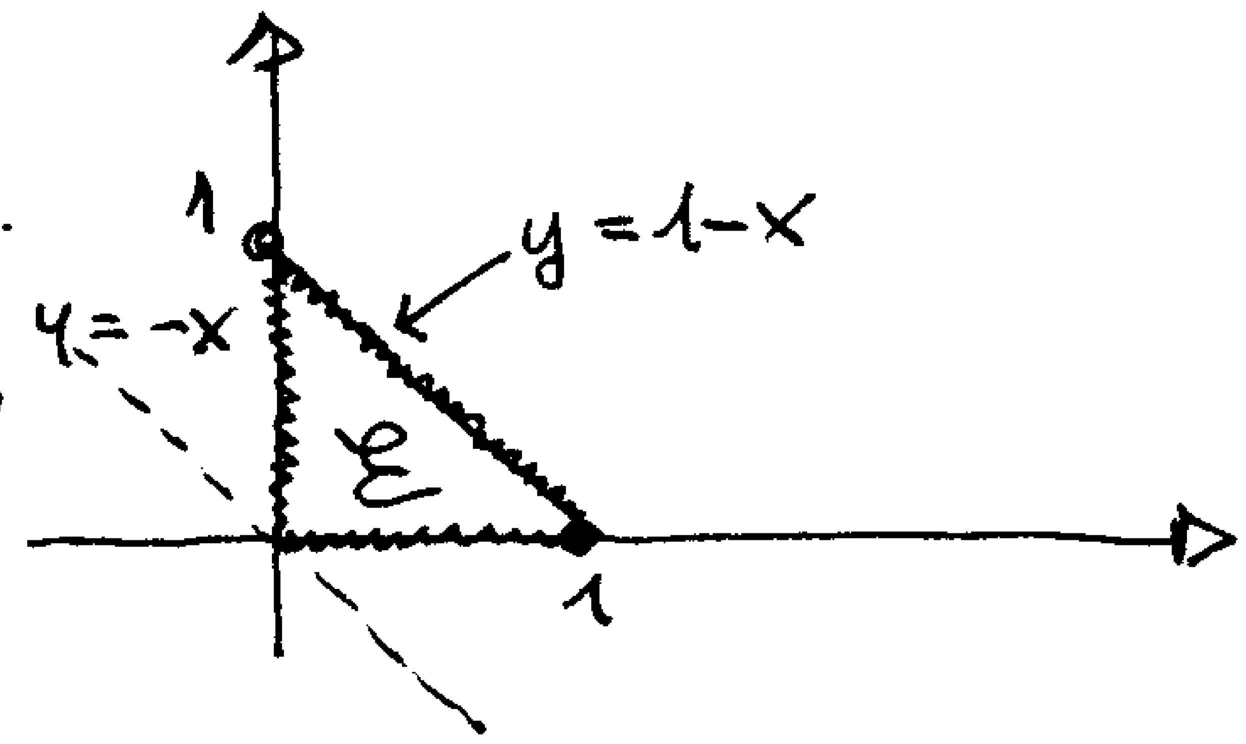
IM4) Dato che: $\dfrac{\partial(y)}{\partial(t_1;t_2;t_3)} = \dfrac{\partial(y)}{\partial(x_1;x_2)} \cdot \dfrac{\partial(x_1;x_2)}{\partial(t_1;t_2;t_3)}$ si ha:

$$\|1\ \ 3\ \ 4\| = \frac{\partial(y)}{\partial(x_1;x_2)} = \left\| \frac{\partial y}{\partial x_1}\ \ \frac{\partial y}{\partial x_2} \right\| \cdot \left\| \begin{matrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{matrix} \right\| = \left\| 1\cdot y'_1 + 0y'_2 \quad -1y'_1+2y'_2 \quad 2y'_1+1\cdot y'_2 \right\| \Rightarrow$$

$$\Rightarrow \begin{cases} y'_1 = 1 \\ -y'_1 + 2y'_2 = 3 \\ 2y'_1 + y'_2 = 4 \end{cases} \Rightarrow \begin{cases} y'_1 = 1 \\ y'_2 = 4-2 = 2 \\ -1+4 = 3 \text{ vera} \end{cases} . \text{ Quindi } \frac{\partial(y)}{\partial(x_1;x_2)} = \|1\ \ 2\|.$$

IIM1) $f(x;y) = x^3+y^3$: funzione differenziabile.

$\mathcal{E}$ insieme limitato e chiuso, quindi esistono il Massimo ed il minimo.



Non conviene usare Kuhn-Tucker. Cerchiamo Max e min liberi e poi studiamo la restrizione di $f(x;y)$ ai tre lati della frontiera di $\mathcal{E}$.

$$\nabla f = 0 \Rightarrow \begin{cases} f'_x = 3x^2 = 0 \\ f'_y = 3y^2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases};\ H(x;y) = \left\| \begin{matrix} 6x & 0 \\ 0 & 6y \end{matrix} \right\|;\ H(0;0) = \left\| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\|, \text{ quindi ???.}$$

Dato che $x^3+y^3 = (x+y)\cdot(x^2-xy+y^2) \geq 0$ per $x+y \geq 0 \Rightarrow y \geq -x$ e dato che $f(0;0) = 0$, relativamente ad $\mathcal{E}$ il punto $(0;0)$ è punto di minimo assoluto.

Se $x = 0 \Rightarrow f(0;y) = y^3 \Rightarrow f'(y) = 3y^2 \geq 0\ \forall y \in [0;1]$

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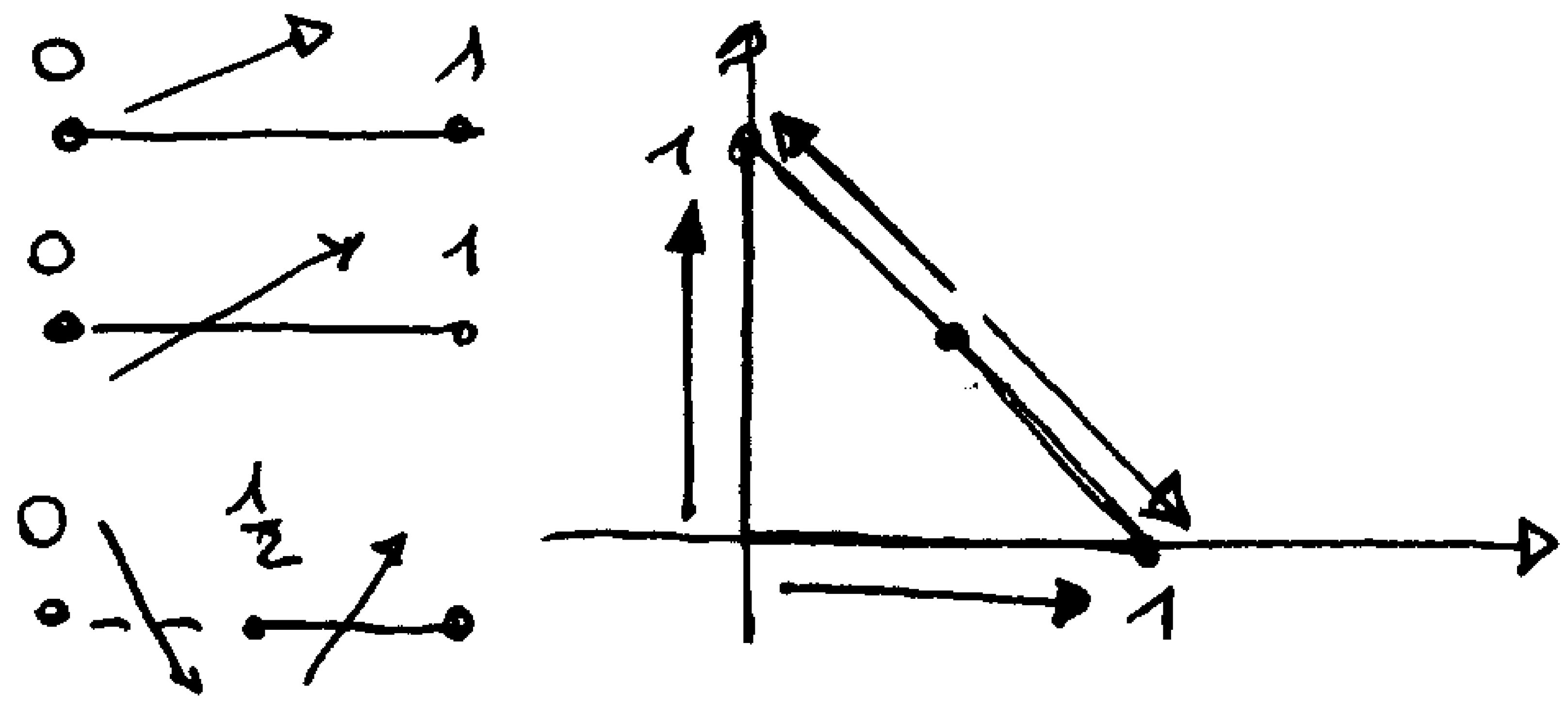
Se $y = 1-x \Rightarrow f(x;1-x) = x^3 + (1-x)^3 \Rightarrow$
$\Rightarrow f'(x) = 3x^2 - 3(1-x)^2 = 3(2x-1) \geq 0$ per $x \geq \frac{1}{2}$



IIM2) $y' = \dfrac{\sqrt{y}}{\sqrt{x}} \Rightarrow \dfrac{1}{\sqrt{y}} \cdot y' = \dfrac{1}{\sqrt{x}} \Rightarrow \displaystyle\int \frac{1}{\sqrt{y}}\,dy = \int \frac{1}{\sqrt{x}}\,dx + K \Rightarrow 2\sqrt{y} = 2\sqrt{x} + K \Rightarrow$

$\Rightarrow \sqrt{y} = \sqrt{x} + m\ \left(m = \frac{K}{2}\right) \Rightarrow y = (\sqrt{x}+m)^2.\ y(1) = 1 \Rightarrow 1 = (1+m)^2 \Rightarrow m = 0.$

Soluzione del problema: $y = (\sqrt{x})^2 = x$ con $x > 0$. La funzione $y = 0$ è una soluzione del problema; è una soluzione singolare.

$\boxed{CAM3}$

II M3) $\begin{cases} x' = x + y + e^t \\ y' = -x + y - e^t \end{cases} \Rightarrow \begin{cases} x' - x - y = e^t \\ x + y' - y = -e^t \end{cases} \Rightarrow \begin{Vmatrix} D-1 & -1 \\ 1 & D-1 \end{Vmatrix} \cdot \begin{Vmatrix} X \\ Y \end{Vmatrix} = \begin{Vmatrix} e^t \\ -e^t \end{Vmatrix}.$

$\begin{vmatrix} D-1 & -1 \\ 1 & D-1 \end{vmatrix} (X) = \begin{vmatrix} e^t & -1 \\ -e^t & D-1 \end{vmatrix} \Rightarrow (D^2 - 2D + 1 + 1)(X) = (D-1)(e^t) - e^t \Rightarrow X'' - 2X' + 2X = -e^t.$

$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm \sqrt{1-2} = 1 \pm \sqrt{-1} = 1 \pm i \Rightarrow \lambda_1 = 1+i \; ; \lambda_2 = 1 - i.$

Soluzione omogenea: $x(t) = c_1 e^t \cos t + c_2 e^t \operatorname{sen} t$. Per la non omogenea:

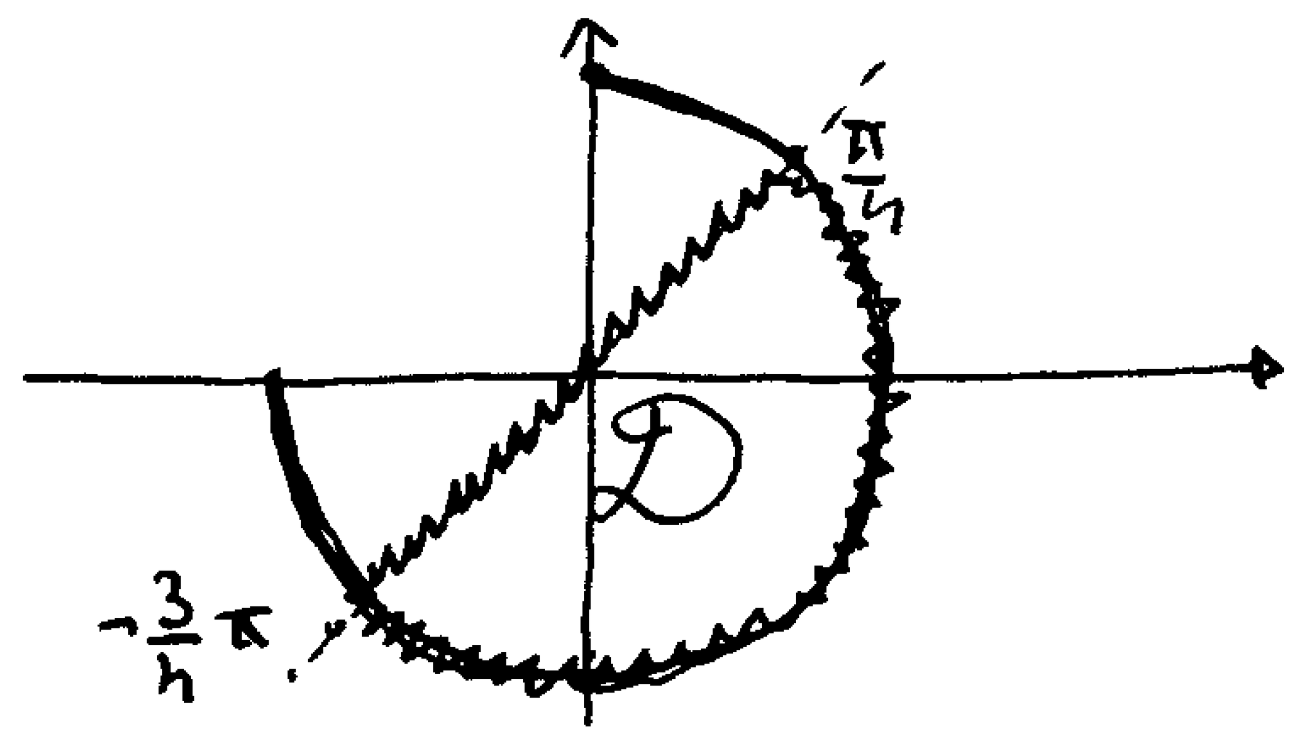
se $x_0 = a \cdot e^t \Rightarrow x_0' = a e^t = x_0'' \Rightarrow a e^t - 2a e^t + 2a e^t = -e^t \Rightarrow a = -1$.

Quindi $x(t) = c_1 e^t \cos t + c_2 e^t \operatorname{sen} t - e^t$. Da $y = x' - x - e^t$ si ha:

$y(t) = (\cancel{c_1 e^t \cos t} - c_1 e^t \operatorname{sen} t + \cancel{c_2 e^t \operatorname{sen} t} + c_2 e^t \cos t - \cancel{e^t}) - (\cancel{c_1 e^t \cos t} + \cancel{c_2 e^t \operatorname{sen} t} - \cancel{e^t}) - e^t \Rightarrow$

$\Rightarrow y(t) = -c_1 e^t \operatorname{sen} t + c_2 e^t \cos t - e^t.$

II M4) $\iint_D (x+y)^2 \, dx \, dy$



Usando le coordinate polari:

$D = \left\{ (\rho ; \vartheta) : 0 \le \rho \le 1 ; -\frac{3}{4}\pi \le \vartheta \le \frac{\pi}{4} \right\}.$

$$\iint_D f(x;y) \, dx \, dy = \int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} \int_0^1 (\rho \cos\vartheta + \rho \operatorname{sen}\vartheta)^2 \cdot \rho \, d\rho \, d\vartheta = \int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} \left(\int_0^1 (\cos\vartheta + \operatorname{sen}\vartheta)^2 \cdot \rho^3 \, d\rho \right) d\vartheta =$$

$$= \int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} (\cos\vartheta + \operatorname{sen}\vartheta)^2 \cdot \left(\frac{1}{4} \rho^4 \Big|_0^1 \right) d\vartheta = \int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} (\operatorname{sen}^2\vartheta + \cos^2\vartheta + 2 \operatorname{sen}\vartheta \cos\vartheta) \cdot \left(\frac{1}{4} - 0 \right) d\vartheta =$$

$$= \frac{1}{4} \int_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} 1 + \operatorname{sen} 2\vartheta \, d\vartheta = \frac{1}{4} \left(\vartheta - \frac{1}{2} \cos 2\vartheta \right|_{-\frac{3}{4}\pi}^{\frac{\pi}{4}} = \frac{1}{4} \left[\left(\frac{\pi}{4} - \frac{1}{2} \cos\frac{\pi}{2} \right) - \left(-\frac{3}{4}\pi - \frac{1}{2} \cos\left(-\frac{3}{2}\pi\right) \right) \right] =$$

$$= \frac{1}{4} \cdot \left[\frac{\pi}{4} - 0 - \left(-\frac{3}{4}\pi - 0 \right) \right] = \frac{1}{4} \cdot \pi .$$