Prova Intermedia di Analisi Matematica del 28/11/2016 $\boxed{\text{PIAM1}}$

IM1) $z = \dfrac{z_1}{z_2 \cdot z_3} = \dfrac{4}{3} \cdot 2 \cdot 3 \cdot \left(\cos\left(\dfrac{4}{3}\pi - \dfrac{1}{6}\pi - \dfrac{2}{3}\pi\right) + i \operatorname{sen}\left(\dfrac{4}{3}\pi - \dfrac{1}{6}\pi - \dfrac{2}{3}\pi\right)\right) =$

$= z = 8 \cdot \left(\cos\dfrac{\pi}{2} + i \operatorname{sen}\dfrac{\pi}{2}\right) = 8i.$

$\sqrt[3]{8i} = \sqrt[3]{8} \cdot \left(\cos\left(\dfrac{\pi}{6} + K\dfrac{2\pi}{3}\right) + i \operatorname{sen}\left(\dfrac{\pi}{6} + K \cdot \dfrac{2\pi}{3}\right)\right); \ 0 \leq K \leq 2.$

Per $K=0$: $2\left(\cos\dfrac{\pi}{6} + i \operatorname{sen}\dfrac{\pi}{6}\right) = 2 \cdot \left(\dfrac{\sqrt{3}}{2} + i\,\dfrac{1}{2}\right) = \sqrt{3} + i$;

Per $K=1$: $2\left(\cos\dfrac{5}{6}\pi + i \operatorname{sen}\dfrac{5}{6}\pi\right) = 2\left(-\dfrac{\sqrt{3}}{2} + i \cdot \dfrac{1}{2}\right) = -\sqrt{3} + i$;

Per $K=2$: $2\left(\cos\dfrac{3}{2}\pi + i \operatorname{sen}\dfrac{3}{2}\pi\right) = 2(-i) = -2i.$

IM2) $f(x;y) = \begin{cases} \dfrac{x^K}{x^2+y^2} & : (x;y) \neq (0;0) \\ 0 & : (x;y) = (0;0) \end{cases}$

Continuità: $\displaystyle\lim_{(x;y)\to(0;0)} \frac{x^K}{x^2+y^2} \Rightarrow \lim_{\rho\to 0} \frac{\rho^K \cdot \cos^K\vartheta}{\rho^2} = 0$ se $K-2>0 \Rightarrow K>2.$

$|\rho^{K-2} \cdot \cos^K\vartheta - 0| \leq \rho^{K-2} \cdot 1 < \varepsilon \Rightarrow \rho < \varepsilon^{\frac{1}{K-2}} \Rightarrow$ Convergenza Uniforme.

La funzione è continua in $(0;0)$ per $K>2$.

Derivate parziali: $\displaystyle\frac{\partial f}{\partial x}(0;0) = \lim_{h\to 0}\left(\frac{(0+h)^K}{(0+h)^2+0^2} - 0\right) \cdot \frac{1}{h} = \lim_{h\to 0} \frac{h^K}{h^3} = \begin{cases} 1 & \text{per } K=3 \\ 0 & \text{per } K>3. \end{cases}$

$\displaystyle\frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0}\left(\frac{0^K}{0^2+(0+h)^2} - 0\right) \cdot \frac{1}{h} = \lim_{h\to 0} \frac{0}{h^3} = \lim_{h\to 0} 0 = 0.$ Vera $\forall K>0$.

Differenziabilità: Per $K=3$: $\displaystyle\lim_{(x;y)\to(0;0)}\left(\frac{x^3}{x^2+y^2} - 0 - (1;0)\cdot(x-0;y-0)\right) \cdot \frac{1}{\sqrt{x^2+y^2}} =$

$\displaystyle= \lim_{(x;y)\to(0;0)}\left(\frac{x^3}{x^2+y^2} - x\right) \cdot \frac{1}{\sqrt{x^2+y^2}} = \lim_{(x;y)\to(0;0)} \frac{-xy^2}{(x^2+y^2)\sqrt{x^2+y^2}} \Rightarrow \lim_{\rho\to 0} \frac{-\rho^3\cos\vartheta \operatorname{sen}^2\vartheta}{\rho^2 \cdot \rho} =$

$\displaystyle= \lim_{\rho\to 0}(-\cos\vartheta \cdot \operatorname{sen}^2\vartheta) = -\cos\vartheta \cdot \operatorname{sen}^2\vartheta \neq 0$ per $\vartheta \neq 0; \frac{\pi}{2}; \pi; \frac{3}{2}\pi.$

La funzione non è differenziabile se $K=3$.

Per $K>3$: $\displaystyle\lim_{(x;y)\to(0;0)}\left(\frac{x^K}{x^2+y^2} - 0 - (0;0)(x-0;y-0)\right) \cdot \frac{1}{\sqrt{x^2+y^2}} = \lim_{(x;y)\to(0;0)} \frac{x^K}{(x^2+y^2)\cdot\sqrt{x^2+y^2}} \Rightarrow$

$\displaystyle\Rightarrow \lim_{\rho\to 0} \frac{\rho^K \cdot \cos^K\vartheta}{\rho^2 \cdot \rho} = \lim_{\rho\to 0} \rho^{K-3} \cdot \cos^K\vartheta = 0$ con Conv. Uniforme per $K>3$.

PIAM2

La funzione è differenziabile per $K > 3$.

IM3) $f(x;y;z) = xyz - xy + xz - yz = 0$. $f(1;1;1) = 1-1+1-1 = 0$.

$\nabla f(x;y;z) = (yz - y + z\,;\, xz - x - z\,;\, xy + x - y)$; $\nabla f(1;1;1) = (1;-1;1)$.

Essendo $f'_z(1;1;1) = 1 \neq 0$ si può definire $(x;y) \to z$.

$\frac{\partial z}{\partial x} = -\frac{1}{1} = -1$; $\frac{\partial z}{\partial y} = -\frac{-1}{1} = 1 \Rightarrow dz(1;1) = z'_x dx + z'_y dy = -1dx + 1dy$.

Da $f(x;y;z) = w = 0 \Rightarrow d^2 w = d^2 f(x;y;z) + f'_z \cdot d^2 z = 0 \Rightarrow d^2 z = -\frac{d^2 f(x;y;z)}{f'_z}$.

$$H(f(x;y;z)) = \begin{Vmatrix} 0 & z-1 & y+1 \\ z-1 & 0 & x-1 \\ y+1 & x-1 & 0 \end{Vmatrix} ; H(f(1;1;1)) = \begin{Vmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{Vmatrix} \Rightarrow d^2 z = -\frac{4\,dx \cdot dz}{1} =$$

$$= -4 \cdot (dx)(-dx + dy) = d^2 z(1;1) = 4(dx)^2 - 4\,dx\,dy.$$

IM4) $\begin{cases} f(x;y;z = e^{x^2+y^2-z^2} - e^{xyz} = 0 \\ g(x;y;z) = xyz + \log\left(\frac{1+x^2+y^2+z^2}{3}\right) = 0 \end{cases} \Rightarrow \begin{cases} f(1;0;1) = e^0 - e^0 = 0 \\ g(1;0;1) = 0 + \log 1 = 0 \end{cases}$.

Poniamo $e^{x^2+y^2-z^2} = Exp \Rightarrow Exp(1;0;1) = 1$.

$$\frac{\partial(f;g)}{\partial(x;y;z)} = \begin{Vmatrix} 2x \cdot Exp - yz \cdot e^{xyz} & 2y \cdot Exp - xz\,e^{xyz} & -2z \cdot Exp - xy\,e^{xyz} \\ yz + \frac{2x}{1+x^2+y^2+z^2} & xz + \frac{2y}{1+x^2+y^2+z^2} & xy + \frac{2z}{1+x^2+y^2+z^2} \end{Vmatrix}$$

$$\frac{\partial(f;g)}{\partial(x;y;z)}(1;0;1) = \begin{Vmatrix} 2-0 & 0-1 & -2-0 \\ 0+\frac{2}{3} & 1+0 & 0+\frac{2}{3} \end{Vmatrix} = \begin{Vmatrix} 2 & -1 & -2 \\ \frac{2}{3} & 1 & \frac{2}{3} \end{Vmatrix}.$$

Essendo $\begin{vmatrix} 2 & -2 \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix} = \frac{4}{3} + \frac{4}{3} \neq 0$ si può definire $y \to (x(y); z(y))$.

$$\frac{dx}{dy} = -\frac{\begin{vmatrix} -1 & -2 \\ 1 & \frac{2}{3} \end{vmatrix}}{\begin{vmatrix} 2 & -2 \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix}} = -\frac{\frac{4}{3}}{\frac{8}{3}} = -\frac{1}{2} ; \frac{dz}{dy} = -\frac{\begin{vmatrix} 2 & -1 \\ \frac{2}{3} & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -2 \\ \frac{2}{3} & \frac{2}{3} \end{vmatrix}} = -\frac{\frac{8}{3}}{\frac{8}{3}} = -1.$$

Equazione retta tangente in $y=0$: $(1;1) + y\cdot\left(-\frac{1}{2};-1\right)$: $y \to \left(1 - \frac{1}{2}y ; 1 - y\right)$.

PIAM3

IM5) $f(x;y) = (x+y)\cdot e^{-\frac{1}{2}(x^2+y^2)}$: Funzione differenziabile 2 volte.

Posto $e^{-\frac{1}{2}(x^2+y^2)} = \mathcal{E}xp$ otteniamo:

$\nabla f(x;y) = \left(1\cdot \mathcal{E}xp + (x+y)(-x)\cdot \mathcal{E}xp \; ; \; 1\cdot \mathcal{E}xp + (x+y)(-y)\cdot \mathcal{E}xp\right) \Rightarrow$

$\nabla f(x;y) = \left((1-x^2-xy)\cdot \mathcal{E}xp \; ; \; (1-xy-y^2)\cdot \mathcal{E}xp\right)$. $\nabla f(1;1) = \left(-1\cdot e^{-1}; -1\cdot e^{-1}\right) \Rightarrow$

$\nabla f(1;1) = \left(-\frac{1}{e}; -\frac{1}{e}\right) \Rightarrow \mathcal{D}_v f(1;1) = \nabla f(1;1)(\cos\alpha; \operatorname{sen}\alpha) = \left(-\frac{1}{e}\right)(1;1)(\cos\alpha; \operatorname{sen}\alpha) \Rightarrow$

$\Rightarrow \mathcal{D}_v f(1;1) = -\frac{1}{e}(\cos\alpha + \operatorname{sen}\alpha)$.

$f''_{xx} = (-2x-y)\cdot \mathcal{E}xp + (1-x^2-xy)\cdot(-x)\cdot \mathcal{E}xp = (x^3+x^2y-3x-y)\cdot \mathcal{E}xp$;

$f''_{xy} = (-x)\cdot \mathcal{E}xp + (1-x^2-xy)\cdot(-y)\cdot \mathcal{E}xp = (x^2y+xy^2-x-y)\cdot \mathcal{E}xp = f''_{yx}$;

$f''_{yy} = (-x-2y)\cdot \mathcal{E}xp + (1-xy-y^2)\cdot(-y)\cdot \mathcal{E}xp = (xy^2+y^3-x-3y)\cdot \mathcal{E}xp$.

$$H(f(1;1)) = \begin{Vmatrix} -2\cdot e^{-1} & 0\cdot e^{-1} \\ 0\cdot e^{-1} & -2\,e^{-1} \end{Vmatrix} = \begin{Vmatrix} -\frac{2}{e} & 0 \\ 0 & -\frac{2}{e} \end{Vmatrix}.$$

$$\mathcal{D}^2_{v,w}(1;1) = \begin{Vmatrix} \cos\alpha & \operatorname{sen}\alpha \end{Vmatrix} \cdot \begin{Vmatrix} -\frac{2}{e} & 0 \\ 0 & -\frac{2}{e} \end{Vmatrix} \cdot \begin{Vmatrix} -\cos\alpha \\ \operatorname{sen}\alpha \end{Vmatrix} = (\cos\alpha; \operatorname{sen}\alpha)\cdot\left(\frac{2}{e}\cos\alpha; -\frac{2}{e}\operatorname{sen}\alpha\right) =$$

$= \frac{2}{e}(\cos^2\alpha - \operatorname{sen}^2\alpha) = \frac{2}{e}(\cos\alpha + \operatorname{sen}\alpha)\cdot(\cos\alpha - \operatorname{sen}\alpha)$.

$\mathcal{D}_v f(1;1) = \mathcal{D}^2_{v,w} f(1;1) \Rightarrow -\frac{1}{e}(\cos\alpha + \operatorname{sen}\alpha) = \frac{2}{e}(\cos\alpha + \operatorname{sen}\alpha)(\cos\alpha - \operatorname{sen}\alpha) \Rightarrow$

$\Rightarrow (\cos\alpha + \operatorname{sen}\alpha)\left(2(\cos\alpha - \operatorname{sen}\alpha) + 1\right) = 0$.

$\cos\alpha + \operatorname{sen}\alpha = 0 \Rightarrow \cos\alpha = -\operatorname{sen}\alpha \Rightarrow \alpha = \frac{3}{4}\pi$ e $\alpha = \frac{7}{4}\pi$.

$2\cos\alpha - 2\operatorname{sen}\alpha + 1 = 0 \Rightarrow \left(\operatorname{sen}\alpha = \dfrac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} = \dfrac{2t}{1+t^2} ; \cos\alpha = \dfrac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} = \dfrac{1-t^2}{1+t^2}\right) \Rightarrow$

$\Rightarrow 2\cdot\dfrac{1-t^2}{1+t^2} - 2\cdot\dfrac{2t}{1+t^2} + 1 = \dfrac{2-2t^2-4t+1+t^2}{1+t^2} = 0 \Rightarrow t^2+4t-3 = 0 \Rightarrow$

$\Rightarrow t = -2 \pm \sqrt{4+3} = -2 \pm \sqrt{7}$. $\operatorname{tg}\frac{\alpha}{2} = t = -2 \pm \sqrt{7} \Rightarrow \frac{\alpha}{2} = \operatorname{arctg}(-2 \pm \sqrt{7})$

$\Rightarrow \alpha = 2\cdot \operatorname{arctg}(-2 \pm \sqrt{7})$.