Prova Intermedia di Mathematics for Economic Applications del 17/12/2010

IM1) $\frac{1}{3}\left(\frac{4}{1+i}-\frac{5}{2-i}\right)=\frac{1}{3}\cdot\frac{8-4i-5-5i}{(1+i)(2-i)}=\frac{1}{3}\;\frac{3-9i}{3+i}\cdot\frac{3-i}{3-i}=\frac{1}{3}\cdot\frac{9-9-30i}{9+1}=-i.$

$-i=\cos\left(\frac{3}{2}\pi\right)+i\,\mathrm{sen}\left(\frac{3}{2}\pi\right).$

$\sqrt[3]{-i}=\cos\left(\frac{\pi}{2}+k\cdot\frac{2\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{2}+k\cdot\frac{2\pi}{3}\right);\ 0\le k\le 2.$

Per $k=0$: $\cos\frac{\pi}{2}+i\,\mathrm{sen}\frac{\pi}{2}=i$; per $k=1$: $\cos\frac{7}{6}\pi+i\,\mathrm{sen}\frac{7}{6}\pi=-\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}$;

per $k=2$: $\cos\frac{11}{6}\pi+i\,\mathrm{sen}\frac{11}{6}\pi=\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}$.

$z_1\cdot z_2\cdot z_3=\cos\left(\frac{\pi}{2}+\frac{7}{6}\pi+\frac{11}{6}\pi\right)+i\,\mathrm{sen}\left(\frac{\pi}{2}+\frac{7}{6}\pi+\frac{11}{6}\pi\right)=\cos\frac{21}{6}\pi+i\,\mathrm{sen}\frac{21}{6}\pi=$

$=\cos\frac{7}{2}\pi+i\,\mathrm{sen}\frac{7}{2}\pi=\cos\frac{3}{2}\pi+i\,\mathrm{sen}\frac{3}{2}\pi=-i.$

IM2) $A=\begin{Vmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&1\end{Vmatrix}\rightarrow\begin{vmatrix}1-\lambda&0&0&1\\0&1-\lambda&0&0\\0&0&1-\lambda&0\\1&0&0&1-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}1-\lambda&0&1\\0&1-\lambda&0\\1&0&1-\lambda\end{vmatrix}=(1-\lambda)^2\begin{vmatrix}1-\lambda&1\\1&1-\lambda\end{vmatrix}=$

$=(1-\lambda)^2\left((1-\lambda)^2-1\right)=(1-\lambda)^2(\lambda^2-2\lambda)=\lambda\cdot(1-\lambda)^2(\lambda-2)=0\Rightarrow\lambda_1=0;\lambda_2=\lambda_3=1;\lambda_4=2.$

Per $\lambda=0$: $\|A-0\mathbb{I}\|=\|A\|\cdot X=\underline{0}\Rightarrow\begin{cases}x_1+x_4=0\\x_2=0\\x_3=0\end{cases}\Rightarrow\begin{cases}x_4=-x_1\\x_2=0\\x_3=0\end{cases}$: Autovettori: $(1;0;0;-1)$

Per $\lambda=1$: $\|A-1\mathbb{I}\|\cdot X=\underline{0}\Rightarrow\begin{cases}x_4=0\\x_1=0\end{cases}\Rightarrow(0;x_2;x_3;0)\to(0;1;0;0),\ (0;0;1;0)$

Per $\lambda=2$: $\|A-2\mathbb{I}\|\cdot X=\underline{0}\Rightarrow\begin{cases}-x_1+x_4=0\\-x_2=0\\-x_3=0\end{cases}\Rightarrow\begin{cases}x_4=x_1\\x_2=0\\x_3=0\end{cases}$: $(1;0;0;1)$.

Autoversori: $(1;0;0;-1)\to\left(\frac{1}{\sqrt2};0;0;-\frac{1}{\sqrt2}\right)$; $(1;0;0;1)\to\left(\frac{1}{\sqrt2};0;0;\frac{1}{\sqrt2}\right)$.

$U=\begin{Vmatrix}\frac{1}{\sqrt2}&0&0&\frac{1}{\sqrt2}\\0&1&0&0\\0&0&1&0\\-\frac{1}{\sqrt2}&0&0&\frac{1}{\sqrt2}\end{Vmatrix}$; $U^T\cdot A\cdot U=\begin{Vmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&2\end{Vmatrix}$.

IM3) $\left\Vert\begin{array}{ccc|c}1&2&1&1\\-2&1&3&1\\2&-1&k&m\end{array}\right\Vert\xrightarrow[R_3\leftarrow R_3-2R_1]{R_2\leftarrow R_2+2R_1}\left\Vert\begin{array}{ccc|c}1&2&1&1\\0&5&5&3\\0&-5&k-2&m-2\end{array}\right\Vert\xrightarrow{R_3\leftarrow R_3+R_2}\left\Vert\begin{array}{ccc|c}1&2&1&1\\0&5&5&3\\0&0&k+3&m+1\end{array}\right\Vert$

Se $k+3=0\Rightarrow k=-3$ e $m+1=0\Rightarrow m=-1$: $\mathrm{Car}(A)=2=\mathrm{Car}(A|Y)$: ∞^1 soluzioni.

Se $k+3\neq0$; $k\neq-3$; $\forall m$: $\mathrm{Car}(A)=3=\mathrm{Car}(A|Y)$: 1 sola soluzione.

Se $k=-3$ e $m+1\neq0$: $m\neq-1$: $\mathrm{Car}(A)=2<\mathrm{Car}(A|Y)=3$: nessuna soluzione.

Se $k=-3$ i tre vettori sono un insieme di generatori di un sottospazio di dimensione 2; se $k\neq-3$ i tre vettori sono una base di $\mathbb{R}^3$.

IM4) $\begin{Vmatrix} 1 & 2 \\ 3 & 4 \end{Vmatrix} \cdot \begin{Vmatrix} x & y \\ z & w \end{Vmatrix} = \begin{Vmatrix} x & y \\ z & w \end{Vmatrix} \cdot \begin{Vmatrix} 3 & 2 \\ 4 & 2 \end{Vmatrix}$: $A \cdot P = P \cdot B$ P.I.M.f.E.A.2

$$\begin{cases} x+2z = 3x+4y \\ y+2w = 2x+2y \\ 3x+4z = 3z+4w \\ 3y+4w = 2z+2w \end{cases} \Rightarrow \begin{cases} 2x = 2z-4y \\ y = 2w-2x \\ z = 4w-3x \\ 2w = 2z-3y \end{cases} \Rightarrow \begin{cases} x = z-2y = z-\frac{4}{3}z+\frac{4}{3}w = \frac{4}{3}w-\frac{1}{3}z \\ y = 2w-2z+4y \Rightarrow y = \frac{2}{3}z-\frac{2}{3}w \\ z = 4w-3z+4z-4w \Rightarrow 0=0 \\ 2w = 2z-2z+2w \Rightarrow 0=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{Vmatrix} \frac{4}{3}w-\frac{1}{3}z & \frac{2}{3}z-\frac{2}{3}w \\ z & w \end{Vmatrix} = P. \quad \text{Se } z=w=3 \Rightarrow \begin{Vmatrix} 3 & 0 \\ 3 & 3 \end{Vmatrix} = P.$$

IM5) $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & k \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -k & k \end{vmatrix} = k \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = k \neq 0$; se $k \neq 0$ esiste A^{-1}.

Se $|A^{-1}| = \frac{1}{3} \Rightarrow |A| = k = 3$.

$$\left\Vert\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right\Vert \to \left\Vert\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right\Vert \to \left\Vert\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right\Vert \to$$

$$\to \left\Vert\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right\Vert \to \left\Vert\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right\Vert \to \left\Vert\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right\Vert.$$

Quindi $A^{-1} = \begin{Vmatrix} 1 & -1 & 0 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{Vmatrix}$.