Compito di Mathematics for economic applications del 17/1/2011 [MfEA1]

IM1) $z = 1-\sqrt{3}\,i = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{5}{3}\pi + i\,\text{sen}\,\frac{5}{3}\pi\right)$

$z^{\frac{3}{2}} = \sqrt{z^3} \Rightarrow z^3 = 2^3\cdot\left(\cos 3\cdot\frac{5}{3}\pi + i\,\text{sen}\,3\cdot\frac{5}{3}\pi\right) = 8(\cos 5\pi + i\,\text{sen}\,5\pi) = 8\cdot(\cos\pi + i\,\text{sen}\,\pi) = -8.$

$\sqrt{-8} = \sqrt{8}\cdot\left(\cos\left(\frac{\pi}{2} + k\cdot\frac{2\pi}{2}\right) + i\,\text{sen}\left(\frac{\pi}{2} + k\cdot\frac{2\pi}{2}\right)\right);\quad 0\le k\le 1.$

per $k=0$: $2\sqrt{2}\left(\cos\frac{\pi}{2} + i\,\text{sen}\,\frac{\pi}{2}\right) = 2\sqrt{2}\,i$

per $k=1$: $2\sqrt{2}\left(\cos\frac{3}{2}\pi + i\,\text{sen}\,\frac{3}{2}\pi\right) = -2\sqrt{2}\,i.$

IM2) $A\cdot B = \begin{Vmatrix} 2 & -2 \\ 3 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} 0 & 1 \\ 2 & 1 \end{Vmatrix} = \begin{Vmatrix} -4 & 0 \\ 2 & 4 \end{Vmatrix} \to \begin{vmatrix} -4-\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)(-4-\lambda) = 0 \Rightarrow \lambda_1 = 4;\ \lambda_2 = -4;$ Gli autovalori sono uguali.

$B\cdot A = \begin{Vmatrix} 0 & 1 \\ 2 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} 2 & -2 \\ 3 & 1 \end{Vmatrix} = \begin{Vmatrix} 3 & 1 \\ 7 & -3 \end{Vmatrix} \to \begin{vmatrix} 3-\lambda & 1 \\ 7 & -3-\lambda \end{vmatrix} = \lambda^2 - 9 - 7 = \lambda^2 - 16 = 0 \Rightarrow \lambda_1 = 4;\ \lambda_2 = -4.$

$\|A\cdot B - 4\mathbb{I}\| \Rightarrow \begin{Vmatrix} -8 & 0 \\ 2 & 0 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow 2x = 0$: Autovettore $(0;1)$

$\|A\cdot B - (-4)\mathbb{I}\| \Rightarrow \begin{Vmatrix} 0 & 0 \\ 2 & 8 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow 2x + 8y = 0 \Rightarrow x = -4y$: Autovettore $(-4;1)$

$P = \begin{Vmatrix} 0 & -4 \\ 1 & 1 \end{Vmatrix}$

$\|B\cdot A - 4\mathbb{I}\| \Rightarrow \begin{Vmatrix} -1 & 1 \\ 7 & -7 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow -x + y = 0 \Rightarrow y = x$: Autovettore $(1;1)$

$\|B\cdot A - (-4)\mathbb{I}\| \Rightarrow \begin{Vmatrix} 7 & 1 \\ 7 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow 7x + y = 0 \Rightarrow y = -7x$: Autovettore $(1;-7)$

$P = \begin{Vmatrix} 1 & 1 \\ 1 & -7 \end{Vmatrix}$

Le matrici modali che diagonalizzano $A\cdot B$ e $B\cdot A$ sono diverse.

IM3) $A\cdot X = Y \Rightarrow X = A^{-1}\cdot Y \Rightarrow \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} 1 \\ -1 \\ 2 \end{Vmatrix}.\quad \begin{vmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = 1\cdot(1+4) = 5 \neq 0.$

$\begin{Vmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{Vmatrix} \xrightarrow{AGG} \begin{Vmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ 2 & 0 & 1 \end{Vmatrix} \xrightarrow{T} \begin{Vmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ -2 & 0 & 1 \end{Vmatrix} \Rightarrow A^{-1} = \begin{Vmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{Vmatrix}.$

$X = \begin{Vmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{Vmatrix}\cdot\begin{Vmatrix} 1 \\ -1 \\ 2 \end{Vmatrix} = \begin{Vmatrix} \frac{1}{5} + 0 + \frac{4}{5} \\ 0 - 1 + 0 \\ -\frac{2}{5} + 0 + \frac{2}{5} \end{Vmatrix} = \begin{Vmatrix} 1 \\ -1 \\ 0 \end{Vmatrix}.$

IM4) $A = \begin{Vmatrix} 1 & 2 & 4 \\ 1 & 0 & 1 \\ 2 & k & 1 \\ 1 & 2 & k \end{Vmatrix} \to \begin{Vmatrix} 1 & 2 & 4 \\ 0 & -2 & -3 \\ 0 & k-4 & -7 \\ 0 & 0 & k-4 \end{Vmatrix}.$

Se $k=4$: $\begin{Vmatrix} 1 & 2 & 4 \\ 0 & -2 & -3 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{Vmatrix}$: $\text{Car}(A) = 3 \Rightarrow \begin{cases} \text{Dim}(\text{Imm}) = 3 \\ \text{Dim}(\text{Nucleo}) = 3-3 = 0 \end{cases}$

Se $k\neq 4$: $\begin{vmatrix} 1 & 2 & 4 \\ 0 & -2 & -3 \\ 0 & 0 & k-4 \end{vmatrix} \neq 0 \Rightarrow \text{Car}(A) = 3 \Rightarrow \begin{cases} \text{Dim}(\text{Imm}) = 3 \\ \text{Dim}(\text{Nucleo}) = 3-3 = 0. \end{cases}$

Quindi $\forall k\in\mathbb{R}$: $\text{Car}(A) = 3.$

II M1) $f(x;y) = x^4 y - y^3 x = 0 \quad P=(1;1) \quad f(P)=0$ $\boxed{MfEA2}$

$\nabla f(x;y) = (4x^3 y - y^3 ; x^4 - 3y^2 x) \; ; \; \nabla f(1;1) = (3;-2)$

$y'(1) = -\frac{3}{-2} = \frac{3}{2} \; ; \quad H(f(x;y)) = \begin{Vmatrix} 12x^2 y & 4x^3 - 3y^2 \\ 4x^3 - 3y^2 & -6xy \end{Vmatrix} \; ; \; H(f(1;1)) = \begin{Vmatrix} 12 & 1 \\ 1 & -6 \end{Vmatrix}.$

$y''(1) = -\dfrac{12 + 2(1)\cdot\frac{3}{2} + (-6)\frac{9}{4}}{-2} = \frac{3}{4} \; : \; P_2(x;1) = 1 + \frac{3}{2}(x-1) + \frac{1}{2}\cdot\frac{3}{4}\cdot(x-1)^2.$

II M2) $\begin{cases} \text{Max/min } f(x;y) = x+y-3 \\ \text{S.V.}: \; xy - x = 1 \end{cases}$

$\Lambda = x+y-3 - \lambda(xy - x - 1)$

$\begin{cases} \Lambda'_x = 1 - \lambda y + \lambda = 0 \\ \Lambda'_y = 1 - \lambda x = 0 \\ xy - x = 1 \end{cases} \Rightarrow \begin{cases} y = \frac{1+\lambda}{\lambda} = 1 + \frac{1}{\lambda} \\ x = \frac{1}{\lambda} \\ \frac{1}{\lambda}\left(1 + \frac{1}{\lambda}\right) - \frac{1}{\lambda} = \frac{1}{\lambda}\left(1 + \frac{1}{\lambda} - 1\right) = \frac{1}{\lambda^2} = 1 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1 \end{cases}$

Se $\lambda = 1 \Rightarrow x = 1$ e $y = 2 \Rightarrow P_1 = (1;2;1)$

Se $\lambda = -1 \Rightarrow x = -1$ e $y = 0 \Rightarrow P_2 = (-1;0;-1)$

$\overline{H}(x;y;\lambda) = \begin{Vmatrix} 0 & g'_x & g'_y \\ g'_x & \Lambda''_{xx} & \Lambda''_{xy} \\ g'_y & \Lambda''_{yx} & \Lambda''_{yy} \end{Vmatrix} = \begin{Vmatrix} 0 & y-1 & x \\ y-1 & 0 & -\lambda \\ x & -\lambda & 0 \end{Vmatrix}$

$\left|\overline{H}_3(1;2;1)\right| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = 1\cdot(-1-1) = -2 < 0$: Punto di minimo

$\left|\overline{H}_3(-1;0;-1)\right| = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -1 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (-1)(-1-1) = 2 > 0$: Punto di Massimo.

II M3) $f(x;y) = e^{x-y} - e^{y-x}$: funzione continua con derivate parziali continue, quindi $\mathcal{D}_v f(1;1) = \nabla f(1;1) \cdot v$.

$\nabla f(x;y) = (e^{x-y} + e^{y-x} ; -e^{x-y} - e^{y-x}) \; ; \; \nabla f(1;1) = (2;-2)$

$\| 1; -1 \| = \sqrt{2} \Rightarrow v = \left(\frac{1}{\sqrt{2}} ; -\frac{1}{\sqrt{2}}\right)$

$\mathcal{D}_v f(1;1) = (2;-2)\cdot\left(\frac{1}{\sqrt{2}} ; -\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}.$

II M4) $f(x;y;z) = x^2z^2 + y^2 - z^2 - xy$

$$\begin{cases} f'_x = 2xz^2 - y = 0 \\ f'_y = 2y - x = 0 \\ f'_z = 2x^2z - 2z = 2z(x^2-1) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \\ z = 0 \end{cases} \cup \begin{cases} z^2 = \frac{y}{2x} \\ y = \frac{x}{2} \\ x = \pm 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 1 \\ y = \frac{1}{2} \\ z^2 = \frac{1}{4} \Rightarrow z = \pm\frac{1}{2} \end{cases} \cup \begin{cases} x = -1 \\ y = -\frac{1}{2} \\ z^2 = \frac{1}{4} \Rightarrow z = \pm\frac{1}{2} \end{cases}$$

$P_1 = (1; \frac{1}{2}; \frac{1}{2})$; $P_2 = (1; \frac{1}{2}; -\frac{1}{2})$; $P_3 = (-1; -\frac{1}{2}; \frac{1}{2})$; $P_4 = (-1; -\frac{1}{2}; -\frac{1}{2})$

$$H(x;y;z) = \begin{Vmatrix} 2z^2 & -1 & 4xz \\ -1 & 2 & 0 \\ 4xz & 0 & 2x^2-2 \end{Vmatrix} ; H(0;0;0) = \begin{Vmatrix} 0 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{Vmatrix} : \begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix} = -1 < 0 : \text{Sella};$$

$$H(1;\tfrac{1}{2};\tfrac{1}{2}) = \begin{Vmatrix} \frac{1}{2} & -1 & 2 \\ -1 & 2 & 0 \\ 2 & 0 & 0 \end{Vmatrix} : \begin{vmatrix} \frac{1}{2} & -1 \\ -1 & 2 \end{vmatrix} = 0 ; \begin{vmatrix} \frac{1}{2} & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0 : \text{Sella};$$

$$H(1;\tfrac{1}{2};-\tfrac{1}{2}) = \begin{Vmatrix} \frac{1}{2} & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 0 \end{Vmatrix} : \begin{vmatrix} \frac{1}{2} & -1 \\ -1 & 2 \end{vmatrix} = 0 ; \begin{vmatrix} \frac{1}{2} & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0 : \text{Sella};$$

$$H(-1;-\tfrac{1}{2};\tfrac{1}{2}) = \begin{Vmatrix} \frac{1}{2} & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 0 \end{Vmatrix} : \begin{vmatrix} \frac{1}{2} & -1 \\ -1 & 2 \end{vmatrix} = 0 ; \begin{vmatrix} \frac{1}{2} & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0 : \text{Sella};$$

$$H(-1;-\tfrac{1}{2};-\tfrac{1}{2}) = \begin{Vmatrix} \frac{1}{2} & -1 & 2 \\ -1 & 2 & 0 \\ 2 & 0 & 0 \end{Vmatrix} : \begin{vmatrix} \frac{1}{2} & -1 \\ -1 & 2 \end{vmatrix} = 0 ; \begin{vmatrix} \frac{1}{2} & 2 \\ 2 & 0 \end{vmatrix} = -4 < 0 : \text{Sella}.$$

In ciascuno dei 4 punti abbiamo un minore di guida del I ordine e un minore di guida del II ordine uguali a zero; quindi $d^2f(P)$ è una forma semidefinita. Essendoci un minore principale di ordine II minore di zero si deduce che sono tutti punti di Sella.