Midterm test of Mathematics for Economic Applications 16/12/2011 $\boxed{\text{MfEA1}}$

IM1) $\frac{1-2i}{3-i}+\frac{1+2i}{1-i}=\frac{(1-2i)(1-i)+(1+2i)(3-i)}{(3-i)(1-i)}=\frac{1-i-2i-\cancel{2}+3-i+6i+\cancel{2}}{3-3i-i-1}=$

$=\frac{4+2i}{2-4i}\cdot\frac{2+4i}{2+4i}=\frac{8+4i+16i-8}{4+16}=\frac{20i}{20}=i.$

$i=1\cdot\left(\cos\frac{\pi}{2}+i\,\mathrm{sen}\,\frac{\pi}{2}\right).$

$\sqrt[3]{i}=\sqrt[3]{1}\cdot\left(\cos\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{3}\right)\right);\ 0\le k\le 2.$

$k=0:\ \cos\frac{\pi}{6}+i\,\mathrm{sen}\,\frac{\pi}{6}=\frac{\sqrt3}{2}+i\cdot\frac12;$

$k=1:\ \cos\frac56\pi+i\,\mathrm{sen}\,\frac56\pi=-\frac{\sqrt3}{2}+i\cdot\frac12;$

$k=2:\ \cos\frac32\pi+i\,\mathrm{sen}\frac32\pi=-i.$

IM2) $|A-\lambda\mathbb{I}|=\begin{vmatrix}-\lambda&1&0\\1&-\lambda&1\\0&1&-\lambda\end{vmatrix}=\begin{vmatrix}-\lambda&1&0\\0&-\lambda&1\\\lambda&1&-\lambda\end{vmatrix}=(-\lambda)(\lambda^2-1)+\lambda\cdot1=(-\lambda)\cdot(\lambda^2-2)=0.$

$\lambda_1=0;\ \lambda_2=\sqrt2;\ \lambda_3=-\sqrt2.$

$|B-\lambda\mathbb{I}|=\begin{vmatrix}-\lambda&1&0\\1&-\lambda&-1\\0&-1&-\lambda\end{vmatrix}=\begin{vmatrix}-\lambda&1&0\\0&-\lambda&-1\\-\lambda&-1&-\lambda\end{vmatrix}=(-\lambda)(\lambda^2-1)-\lambda(-1)=(-\lambda)(\lambda^2-2)=0.$

$\lambda_1=0;\ \lambda_2=\sqrt2;\ \lambda_3=-\sqrt2.$

$A\cdot B=\begin{Vmatrix}0&1&0\\1&0&1\\0&1&0\end{Vmatrix}\cdot\begin{Vmatrix}0&1&0\\1&0&-1\\0&-1&0\end{Vmatrix}=\begin{Vmatrix}1&0&-1\\0&0&0\\1&0&-1\end{Vmatrix}\to|A\cdot B-\lambda\mathbb{I}|=\begin{vmatrix}1-\lambda&0&-1\\0&-\lambda&0\\1&0&-1-\lambda\end{vmatrix}=$

$=(-\lambda)\cdot\left((1-\lambda)(-1-\lambda)+1\right)=(-\lambda)\cdot(\lambda^2-1+1)=-\lambda^3=0\Rightarrow\lambda_1=\lambda_2=\lambda_3=0.\ m_0^a=3.$

$\mathrm{RANK}(A\cdot B-0\cdot\mathbb{I})=\mathrm{RANK}(A\cdot B)=1\Rightarrow m_0^g=3-1=2<m_0^a=3.$

So the matrix $A\cdot B$ is not a diagonalizable matrix.

$B\cdot A=\begin{Vmatrix}0&1&0\\1&0&-1\\0&-1&0\end{Vmatrix}\cdot\begin{Vmatrix}0&1&0\\1&0&1\\0&1&0\end{Vmatrix}=\begin{Vmatrix}1&0&1\\0&0&0\\-1&0&-1\end{Vmatrix}=(A\cdot B)^T.$

The matrix $B\cdot A$ is the transpose of the matrix $A\cdot B$.

IM3) $A\sim B\iff A\cdot P=P\cdot B\Rightarrow\begin{Vmatrix}1&2\\2&1\end{Vmatrix}\cdot\begin{Vmatrix}x_1&y_1\\x_2&y_2\end{Vmatrix}=\begin{Vmatrix}x_1&y_1\\x_2&y_2\end{Vmatrix}\cdot\begin{Vmatrix}3&2\\0&-1\end{Vmatrix}\Rightarrow$

$\Rightarrow\begin{cases}x_1+2x_2=3x_1\\y_1+2y_2=2x_1-y_1\\2x_1+x_2=3x_2\\2y_1+y_2=2x_2-y_2\end{cases}\Rightarrow\begin{cases}2x_1=2x_2\\2y_1+2y_2=2x_1\\2x_1=2x_2\\2y_1+2y_2=2x_2\end{cases}\Rightarrow\begin{cases}x_1=x_2\\y_2=x_1-y_1\end{cases}\quad\forall x_1;y_1.$

$P=\begin{Vmatrix}x_1&y_1\\x_1&x_1-y_1\end{Vmatrix}\Rightarrow\begin{Vmatrix}k&m\\k&k-m\end{Vmatrix};\ \begin{vmatrix}k&m\\k&k-m\end{vmatrix}=k^2-mk-mk=k(k-2m)\ne0$

$k\ne0$ and $k\ne2m$.

For $k=1$ and $m=2$: $P=\begin{Vmatrix}1&2\\1&-1\end{Vmatrix}.$

IM4) $\begin{cases} (1;1;-1)\cdot(x;y;z) = x+y-z = 0 \\ (1;-1;0)\cdot(x;y;z) = x-y = 0 \end{cases} \Rightarrow \begin{cases} z = x+y = 2x \\ y = x \end{cases} : (x;x;2x) \to (1;1;2).$ $\boxed{\text{MfEA2}}$

Unit vectors:

$\|(1;1;-1)\| = \sqrt{3} \Rightarrow \left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}};-\frac{1}{\sqrt{3}}\right)$; $\|(1;-1;0)\| = \sqrt{2} \Rightarrow \left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}};0\right)$

$\|(1;1;2)\| = \sqrt{6} \Rightarrow \left(\frac{1}{\sqrt{6}};\frac{1}{\sqrt{6}};\frac{2}{\sqrt{6}}\right).$

Orthonormal basis: $U = \left\{\left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}};-\frac{1}{\sqrt{3}}\right);\left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}};0\right);\left(\frac{1}{\sqrt{6}};\frac{1}{\sqrt{6}};\frac{2}{\sqrt{6}}\right)\right\}.$

For finding Y's coordinates in such a basis:

$$\left\|\begin{matrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \left\|\begin{matrix} 1 \\ -1 \\ 1 \end{matrix}\right\| \Rightarrow \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \left\|\begin{matrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ -1 \\ 1 \end{matrix}\right\| = \left\|\begin{matrix} -\frac{1}{\sqrt{3}} \\ \sqrt{2} \\ \frac{2}{\sqrt{6}} \end{matrix}\right\|$$

as U is an orthogonal matrix, and so $U^{-1} = U^T$.

IM5) $Y = A \cdot X \Rightarrow \left\|\begin{matrix} 1 & -2 & 1 & 3 \\ 2 & 0 & -1 & 2 \\ 0 & -4 & m & K \end{matrix}\right\| \cdot \left\|\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix}\right\| = \left\|\begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix}\right\|.$

Let's compute RANK(A).

$$\left\|\begin{matrix} 1 & -2 & 1 & 3 \\ 2 & 0 & -1 & 2 \\ 0 & -4 & m & K \end{matrix}\right\| \xrightarrow[R_2 \leftarrow R_2 - 2R_1]{} \left\|\begin{matrix} 1 & -2 & 1 & 3 \\ 0 & 4 & -3 & -4 \\ 0 & -4 & m & K \end{matrix}\right\| \xrightarrow[R_3 \leftarrow R_3 + R_2]{} \left\|\begin{matrix} 1 & -2 & 1 & 3 \\ 0 & 4 & -3 & -4 \\ 0 & 0 & m-3 & K-4 \end{matrix}\right\|.$$

If $m=3$ and $K=4$: RANK(A) = 2 → DIM (RANKSPACE) = 2 ; DIM (KERNEL) = 4 − 2 = 2 ;

If $m \neq 3$ or $K \neq 4$: RANK(A) = 3 → DIM (RANKSPACE) = 3 ; DIM (KERNEL) = 4 − 3 = 1 .