Task Mathematics for Economic Applications 31/1/2012

MfEA1

IM1) $x^3-(2+i)x^2+(4+2i)x-4i=0.$

Equation is satisfied if $x=i$:

$$\begin{array}{c|ccc|c} & 1 & -2-i & 4+2i & -4i \\ i & & i & -2i & 4i \\ \hline & 1 & -2 & 4 & 0 \end{array}$$

$x^3-(2+i)x^2+(4+2i)x-4i=(x-i)(x^2-2x+4)=0$

$x=1\pm\sqrt{1-4}=1\pm\sqrt{-3}=1\pm\sqrt{3}\,i.$ $\quad x=1+\sqrt{3}\,i\in I$ Q. of $\mathbb{C}$.

$1+\sqrt{3}\,i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\cdot\left(\cos\frac{\pi}{3}+i\,\mathrm{sen}\,\frac{\pi}{3}\right).$

$\sqrt{1+\sqrt{3}\,i}=\sqrt{2}\cdot\left(\cos\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{2}\right)+i\,\mathrm{sen}\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{2}\right)\right);\ 0\le k\le 1.$

For $k=0$: $\sqrt{2}\left(\cos\frac{\pi}{6}+i\,\mathrm{sen}\,\frac{\pi}{6}\right)=\sqrt{2}\cdot\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=\frac{\sqrt{3}}{\sqrt{2}}+\frac{1}{\sqrt{2}}i;$

For $k=1$: $\sqrt{2}\left(\cos\frac{7}{6}\pi+i\,\mathrm{sen}\,\frac{7}{6}\pi\right)=\sqrt{2}\left(-\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}\right)=-\frac{\sqrt{3}}{\sqrt{2}}-\frac{1}{\sqrt{2}}i.$

IM2) Let $X=\begin{Vmatrix}1&1&-1\\1&2&1\\1&k&k\end{Vmatrix}$; $\{X_1;X_2;X_3\}$ is a base for $\mathbb{R}^3$ iff $|X|\neq 0$.

$$\begin{vmatrix}1&1&-1\\1&2&1\\1&k&k\end{vmatrix}=\begin{vmatrix}1&1&-1\\0&1&2\\0&k-1&k+1\end{vmatrix}=1\cdot\begin{vmatrix}1&2\\k-1&k+1\end{vmatrix}=k+1-2k+2=3-k\neq 0 \text{ if } k\neq 3.$$

$(4;4;4)$ is a linear combination of $X_1;X_2;X_3$ iff $\mathrm{RANK}(X|Y)=\mathrm{RANK}(X)$.

$$\left\Vert\begin{array}{ccc|c}1&1&-1&4\\1&2&1&4\\1&k&k&4\end{array}\right\Vert\to\left\Vert\begin{array}{ccc|c}1&1&-1&4\\0&1&2&0\\0&k-1&k+1&0\end{array}\right\Vert\to\left\Vert\begin{array}{ccc|c}1&1&-1&4\\0&1&2&0\\0&(k-1)-1(k-1)&k+1-2(k-1)&0\end{array}\right\Vert\to$$

$$\to\left\Vert\begin{array}{ccc|c}1&1&-1&4\\0&1&2&0\\0&0&3-k&0\end{array}\right\Vert.$$

If $k\neq 3$: only 1 solution;
If $k=3$: ∞^1 Solutions. There are always solutions.

No values for k to get a system with no solutions.

IM3) $A=\begin{Vmatrix}3&0&1\\-1&2&-1\\1&k&3\end{Vmatrix}\to|A-\lambda\mathbb{I}|=\begin{vmatrix}3-\lambda&0&1\\-1&2-\lambda&-1\\1&k&3-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda&0&1\\0&2-\lambda&-1\\\lambda-2&k&3-\lambda\end{vmatrix}=$

$=(2-\lambda)\big((2-\lambda)(3-\lambda)+k\big)+(\lambda-2)(0-2+\lambda)=(2-\lambda)(\lambda^2-5\lambda+6+k+2-\lambda)=$

$=(2-\lambda)(\lambda^2-6\lambda+8+k)=0.$ $\quad\lambda_1=2$ is the first eigenvalue.

For multiple eigenvalues: $\lambda^2-6\lambda+8+k=0$ if $\lambda=2$ if $4-12+8+k=0\Rightarrow$

$\Rightarrow\boxed{k=0}$: $\quad\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4)=0$ if $\lambda_1=\lambda_2=2$; $\lambda_3=4$.

$\mathrm{RANK}(A-2\mathbb{I})$: $\begin{Vmatrix}1&0&1\\-1&0&-1\\1&0&1\end{Vmatrix}$. $\mathrm{RANK}(A-2\mathbb{I})=1\Rightarrow m_2^g=3-1=2=m_2^a$.

So for $k=0$ the matrix A is diagonalizable.

Now we check for eigenvectors.

MfEA2

For $\lambda = 2$ $(k=0)$

$$\|A-2\mathbb{I}\|\cdot X = \underline{0} \Rightarrow \begin{Vmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow x+z=0 \Rightarrow z=-x, \forall y.$$

Eigenvectors of $\lambda = 2$: $(x; y; -x)$.

For $\lambda = 4$ $(k=0)$

$$\|A-4\mathbb{I}\|\cdot X = \underline{0} \Rightarrow \begin{Vmatrix} -1 & 0 & 1 \\ -1 & -2 & -1 \\ 1 & 0 & -1 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x-z=0 \\ x+2y+z=0 \end{cases} \Rightarrow \begin{cases} z=x \\ y=-x \end{cases}$$

Eigenvectors for $\lambda = 4$: $(x; -x; x) \Rightarrow (1; -1; 1)$.

$(x; y; -x)$ and $(1; -1; 1)$ may be orthogonal vectors?

$(x; y; -x)\cdot(1; -1; 1) = x - y - x = -y = 0$ iff $y = 0$.

So $(x; 0; -x)$ as eigenvector for $\lambda = 2$, but we cannot get from $(x; 0; -x)$ 2 orthogonal vectors. Matrices which diagonalize A cannot be orthogonal matrices.

We can get multiple eigenvalue also if $\lambda^2 - 6\lambda + 8 + k = 0$ has a multiple root: $\lambda = 3 \pm \sqrt{9-8-k} = 3 \pm \sqrt{1-k}$. $\Delta = 0$ iff $\boxed{k=1}$

If $k=1$: $(\lambda-2)(\lambda^2-6\lambda+9) = (\lambda-2)(\lambda-3)^2 = 0 \Rightarrow \lambda_1 = 2;\ \lambda_2 = \lambda_3 = 3$

For $\lambda = 3$ $(k=1)$

$$\text{RANK}(A-3\mathbb{I}) = \text{RANK}\left(\begin{Vmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{Vmatrix}\right) = 2 \Rightarrow m_3^g = 3-2 = 1 < m_3^a = 2.$$

For $k=1$ A is not a diagonalizable matrix.

IM4) The system has solutions if $\text{RANK}(A) = \text{RANK}(A|Y)$.

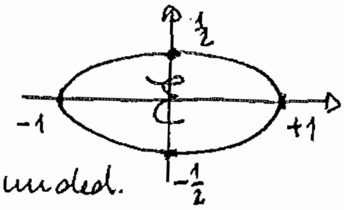
$$\left\|\begin{array}{ccc|c} 1 & 3 & k & 1 \\ 1 & -1 & -2 & 1 \\ 1 & 2 & k & 1 \end{array}\right\| \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - R_1} \left\|\begin{array}{ccc|c} 1 & 3 & k & 1 \\ 0 & -4 & -2-k & 0 \\ 0 & -1 & 0 & 0 \end{array}\right\| \xrightarrow{R_2 \leftrightarrow R_3} \left\|\begin{array}{ccc|c} 1 & 3 & k & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -4 & -2-k & 0 \end{array}\right\| \xrightarrow{R_3 \leftarrow R_3 - 4R_2} \left\|\begin{array}{ccc|c} 1 & 3 & k & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2-k & 0 \end{array}\right\|$$

If $-2-k = 0 \Rightarrow k = -2$: $\text{RANK}(A) = \text{RANK}(A|Y) = 2$: ∞^1 Solutions.

If $-2-k \neq 0 \Rightarrow k \neq -2$: $\text{RANK}(A) = \text{RANK}(A|Y) = 3$: 1 Solution.

The system has always solutions for every value of k.

II M1) $\begin{cases} \text{Max/min } f(x;y) = x^2 - y^3 - 1 \\ \text{s.t.: } x^2 + 4y^2 \le 1 \end{cases}$



MfEA3

f is a continuous function; $\mathcal{E}$ is closed and bounded.
So Weierstrass's Theorem holds. We can apply Kuhn-Tucker Conditions.

$\Lambda = x^2 - y^3 - 1 - \lambda(x^2 + 4y^2 - 1)$.

For $\lambda = 0$

$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = -3y^2 = 0 \\ x^2 + 4y^2 \le 1 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \\ 0 + 0 \le 1 \end{cases}$; $H = \begin{Vmatrix} 2 & 0 \\ 0 & -6y \end{Vmatrix}$; $H(0;0) = \begin{Vmatrix} 2 & 0 \\ 0 & 0 \end{Vmatrix}$: no MAX point.

$f(0;0) = -1$; $f(x;y) - f(0;0) = x^2 - y^3$.

If $y = 0$; $f(x;0) = x^2 > 0 \ \forall x \neq 0$

If $x = 0$ $f(0;y) = -y^3 < 0$ if $y > 0$. So $(0;0)$ is a saddle point.

For $\lambda \neq 0$

$\begin{cases} \Lambda'_x = 2x - 2\lambda x = 2x(1-\lambda) = 0 \\ \Lambda'_y = -3y^2 - 8\lambda y = -y(3y + 8\lambda) = 0 \\ x^2 + 4y^2 = 1 \end{cases}$ $\begin{cases} x = 0 \\ y = 0 \\ 0 + 0 = 1 \end{cases}$ $\cup$ $\begin{cases} \lambda = 1 \\ y = 0 \\ x^2 = 1 \end{cases}$ $\cup$ $\begin{cases} \lambda = 1 \\ y = -\frac{8}{3} \\ x^2 = 1 - 4 \cdot \frac{64}{9} \end{cases}$ $\cup$ $\begin{cases} x = 0 \\ y^2 = \frac{1}{4} \\ \lambda = -\frac{3}{8}y. \end{cases}$

No Solut. (first system); No Solut. (third system)

$\Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda = 1 \end{cases}$ Max? $\begin{cases} x = -1 \\ y = 0 \\ \lambda = 1 \end{cases}$ Max? $\begin{cases} x = 0 \\ y = \frac{1}{2} \\ \lambda = -\frac{3}{16} \end{cases}$ Min? $\begin{cases} x = 0 \\ y = -\frac{1}{2} \\ \lambda = \frac{3}{16} \end{cases}$ Max?

$$\overline{H}(x;y;\lambda) = \begin{Vmatrix} 0 & 2x & 8y \\ 2x & 2-2\lambda & 0 \\ 8y & 0 & -6y-8\lambda \end{Vmatrix}$$

$$|\overline{H}(1;0;1)| = \begin{vmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -8 \end{vmatrix} = (-8)\cdot(-4) > 0 : \underline{\text{Max}} \text{ as for K-T Conditions.}$$

$$|\overline{H}(-1;0;1)| = \begin{vmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -8 \end{vmatrix} = (-8)(-4) > 0 : \underline{\text{Max}} \text{ as for K-T Conditions.}$$

$$|\overline{H}(0;\tfrac{1}{2};-\tfrac{3}{16})| = \begin{vmatrix} 0 & 0 & 4 \\ 0 & \frac{19}{8} & 0 \\ 4 & 0 & -\frac{3}{2} \end{vmatrix} = 4\cdot\left(-4\cdot\frac{19}{8}\right) < 0 : \underline{\text{Min}} \text{ as for K-T Conditions.}$$

$$|\overline{H}(0;-\tfrac{1}{2};\tfrac{3}{16})| = \begin{vmatrix} 0 & 0 & -4 \\ 0 & \frac{13}{8} & 0 \\ -4 & 0 & \frac{3}{2} \end{vmatrix} = (-4)\cdot\left(\frac{13}{2}\right) < 0$$

Min!! different from K-T conditions. So this point is not a maximum nor a minimum point.

$f(1;0) = 1 - 0 - 1 = 0$; $f(-1;0) = 1 - 0 - 1 = 0$. Absolute maximum value.

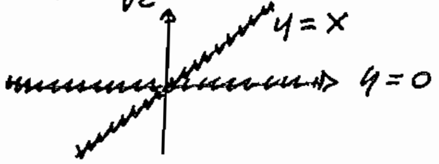
$f(0;\frac{1}{2}) = 0 - \frac{1}{8} - 1 = -\frac{9}{8}$: Absolute minimum value.

II M2) $f(x;y) = y^3 - xy^2 \quad ; \quad \|1;1\| = \sqrt{2} \; ; \; w = \left(\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right)$ MfEA 4

$$\mathcal{D}_w f(x;y) = \nabla f(x;y) \cdot \left(\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right) = \left(-y^2 ; 3y^2 - 2xy\right) \cdot \left(\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right) =$$

$$= \frac{1}{\sqrt{2}}\left(-y^2 + 3y^2 - 2xy\right) = \frac{1}{\sqrt{2}} \cdot \left(2y^2 - 2xy\right) = 0 \Rightarrow \frac{2}{\sqrt{2}} y \cdot (y - x) = 0 \text{ if } \begin{cases} y = 0 \\ y = x \end{cases}.$$

$\mathcal{D}_w f(x;y) = 0$:



II M3) $f(x;y) = x \cdot |x - y^2| \; ; \; f(0;0) = 0 = \lim\limits_{(x;y)\to(0;0)} f(x;y)$: f continuous at $(0;0)$.

$$\frac{\partial f}{\partial x}(0;0) = \lim_{h \to 0} \frac{(0+h)|0+h-0| - 0}{h} = \lim_{h\to 0} \frac{h \cdot |h|}{h} = \lim_{h \to 0} |h| = 0 ;$$

$$\frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0} \frac{0 \cdot |0 - (0+h)^2| - 0}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

f is differentiable at $(0;0)$ if $\lim\limits_{(x;y)\to(0;0)} \dfrac{x \cdot |x - y^2| - 0 - (0;0)(x-0;y-0)}{\sqrt{x^2+y^2}} = 0.$

Using polar cohordinates: $\lim\limits_{\rho \to 0} \dfrac{\rho\cos\vartheta \, |\rho\cos\vartheta - \rho^2 \operatorname{sen}^2\vartheta|}{\rho} =$

$= \lim\limits_{\rho\to 0} \rho \cdot \cos\vartheta \, |\cos\vartheta - \rho \operatorname{sen}^2\vartheta| = 0$. Convergence is uniform; in fact:

$$\left|\rho\cos\vartheta \, |\cos\vartheta - \rho\operatorname{sen}^2\vartheta| - 0\right| \leq \rho \cdot |\cos\vartheta| \cdot |\cos\vartheta - \rho\operatorname{sen}^2\vartheta| \leq \rho \cdot 1 \cdot \left(|\cos\vartheta| + \rho\operatorname{sen}^2\vartheta\right) \leq$$

$$\leq \rho \cdot (1+\rho) < \varepsilon \Rightarrow \rho^2 + \rho - \varepsilon < 0 \text{ if } \rho < \frac{-1+\sqrt{1+4\varepsilon}}{2} = \delta(\varepsilon).$$

So the function is differentiable at $(0;0)$.

II M4) $f(x;y) = x^2 y^3 - e^{x-y} = 0 \; ; \; f(1;1) = 1 - 1 = 0.$

$\nabla f(x;y) = \left(2xy^3 - e^{x-y} ; 3x^2y^2 + e^{x-y}\right) ; \nabla f(1;1) = (1 ; 4).$

Exists $y = y(x)$ as $f'_y = 4 \neq 0$. $\quad y'(1) = -\frac{1}{4}.$

$$H(f(x;y)) = \begin{Vmatrix} 2y^3 - e^{x-y} & 6xy^2 + e^{x-y} \\ 6xy^2 + e^{x-y} & 6x^2y - e^{x-y} \end{Vmatrix} ; H(1;1) = \begin{Vmatrix} 1 & 7 \\ 7 & 5 \end{Vmatrix}.$$

$$y''(1) = -\frac{f''_{xx} + 2f''_{xy} \cdot y' + f''_{yy}(y')^2}{f'_y} = -\frac{1 + 2\cdot 7 \cdot \left(-\frac{1}{4}\right) + 5 \cdot \frac{1}{16}}{4} =$$

$$= -\frac{1 - \frac{7}{2} + \frac{5}{16}}{4} = -\frac{-\frac{35}{16}}{4} = \frac{35}{64}.$$

$$P_2(x;1) = 1 - \frac{1}{4}(x-1) + \frac{1}{2} \cdot \frac{35}{64} \cdot (x-1)^2.$$