

IM1) $\sqrt{(1-i)^3}$. $1-i = \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = \sqrt{2} \left(\cos \frac{7}{4}\pi + i \sin \frac{7}{4}\pi \right)$.

$(1-i)^3 = (\sqrt{2})^3 \cdot \left(\cos 3 \cdot \frac{7}{4}\pi + i \sin 3 \cdot \frac{7}{4}\pi \right) = 2\sqrt{2} \cdot \left(\cos \frac{21}{4}\pi + i \sin \frac{21}{4}\pi \right) =$
 $= 2\sqrt{2} \cdot \left(\cos \frac{5}{4}\pi + i \sin \frac{5}{4}\pi \right) = \left(2\sqrt{2} \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \right)$.

$\sqrt{(1-i)^3} = \sqrt{2\sqrt{2}} \left(\cos \left(\frac{5}{8}\pi + k \cdot \frac{2\pi}{2} \right) + i \sin \left(\frac{5}{8}\pi + k \cdot \frac{2\pi}{2} \right) \right); 0 \leq k \leq 1$.

for $k=0$: $\sqrt{2\sqrt{2}} \left(\cos \frac{5}{8}\pi + i \sin \frac{5}{8}\pi \right)$; for $k=1$: $\sqrt{2\sqrt{2}} \left(\cos \frac{13}{8}\pi + i \sin \frac{13}{8}\pi \right)$.

IM2) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; $|A| = 1$; $\text{Adj}(A) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$; thus $x: \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = A^{-1}$.

$A + A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$: Eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 2$
 Since $A + A^{-1}$ is triangular.

$\text{RANK}((A + A^{-1}) - 2I) = \text{RANK} \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = 1 \Rightarrow m_2^g = 3 - 1 = 2 < 3 = m_2^a$.

$A + A^{-1}$ is not diagonalizable.

$A - A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$: Eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 0$
 Since $A - A^{-1}$ is triangular.

$\text{RANK}((A - A^{-1}) - 0 \cdot I) = \text{RANK}(A - A^{-1}) = 2 \Rightarrow m_0^g = 3 - 2 = 1 < 3 = m_0^a$.

$A - A^{-1}$ is not diagonalizable.

IM3) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & k \end{pmatrix}$. A is a symmetric matrix, so always diagonalizable with an orthogonal matrix, $\forall k \in \mathbb{R}$.

$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & k-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda-2 & 2-k+\lambda \\ 1 & 2 & k-\lambda \end{vmatrix} = (1-\lambda)((-\lambda-2)(k-\lambda) - 2(2-k+\lambda)) + 1(2-k+\lambda + \lambda+2) =$

$= (1-\lambda)(\lambda^2 - k\lambda + 2\lambda - 2k - 4 + 2k - 2\lambda) + 4 - k + 2\lambda =$

$= (1-\lambda)(\lambda^2 - k\lambda - 4) + 4 - k + 2\lambda = \lambda^2 - k\lambda - \lambda^3 + k\lambda^2 + 4\lambda - k + 2\lambda =$

$= -\lambda^3 + (1+k)\lambda^2 + (6-k)\lambda - k = 0 \Rightarrow \lambda^3 - (1+k)\lambda^2 + (k-6)\lambda + k = 0$.

$\lambda = 0$ is an eigenvalue for A if the "known term": k is zero.

So, for $k=0$: $\lambda^3 - \lambda^2 - 6\lambda = \lambda(\lambda^2 - \lambda - 6) = 0$. $\lambda_1 = 0$;

$\lambda = \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm \sqrt{25}}{2} = \frac{1 \pm 5}{2} = \begin{matrix} 3 \\ -2 \end{matrix}$.

if $k=0$: $\lambda_1 = 0$; $\lambda_2 = 3$; $\lambda_3 = -2$.

For $k=0$ and $\lambda=0$: $\|A - 0I\| \cdot X = \underline{0} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0} \Rightarrow \begin{cases} x+y+z=0 \\ x+2z=0 \\ x+2y=0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = -2y \\ z = y \end{cases}$: Eigenvector $(-2y; y; y) \rightarrow (-2; 1; 1) \rightarrow \left(-\frac{2}{\sqrt{6}}; \frac{1}{\sqrt{6}}; \frac{1}{\sqrt{6}}\right)$

$\|V\| = \sqrt{6}$

For $k=0$ and $\lambda=3$: $\|A-3I\| \cdot X = 0 \Rightarrow \begin{vmatrix} -2 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{vmatrix} \cdot \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0 \Rightarrow$

MP EA2

$$\Rightarrow \begin{cases} -2x + y + z = 0 \\ x - 3y + 2z = 0 \\ x + 2y - 3z = 0 \end{cases} \Rightarrow \begin{cases} -2x + y + z = 0 \\ -5y + 5z = 0 \end{cases} \Rightarrow \begin{cases} 2x = 2z \\ y = z \end{cases} \Rightarrow \begin{cases} y = x \\ z = x \end{cases}$$

Eigenvector $(x; x; x) \rightarrow (1; 1; 1) \rightarrow (\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}})$.
 $\|v\| = \sqrt{3}$

For $k=0$ and $\lambda=-2$: $\|A+2I\| \cdot X = 0 \Rightarrow \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{vmatrix} \cdot \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0 \Rightarrow$

$$\Rightarrow \begin{cases} 3x + y + z = 0 \\ x + 2y + 2z = 0 \end{cases} \Rightarrow \begin{cases} -6y - 6z + y + z = 0 \\ x = -2y - 2z \end{cases} \Rightarrow \begin{cases} 5y = -5z \\ x = -2y - 2z \end{cases} \Rightarrow \begin{cases} y = -z \\ x = 0 \end{cases}$$

Eigenvector $(0; y; -y) \rightarrow (0; 1; -1) \rightarrow (0; \frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}})$
 $\|v\| = \sqrt{2}$

Orthogonal matrix which diagonalizes A : $U = \begin{vmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{vmatrix}$.
 $U^T \cdot A \cdot U = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{vmatrix}$.

IM4) $\begin{cases} x_1 - 2x_2 + 2x_3 - x_4 = 0 \\ 2x_1 + 3x_2 - x_3 + 2x_4 = 1 \\ 4x_1 - x_2 + 3x_3 + m x_4 = k \end{cases}$. For Rouché - Capelli Theorem:

$$\left\| \begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 2 & 3 & -1 & 2 & 1 \\ 4 & -1 & 3 & m & k \end{array} \right\| \rightarrow \left\| \begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 0 & 7 & -5 & 4 & 1 \\ 0 & 7 & -5 & m+4 & k \end{array} \right\| \rightarrow$$

$$\left\| \begin{array}{cccc|c} 1 & -2 & 2 & -1 & 0 \\ 0 & 7 & -5 & 4 & 1 \\ 0 & 0 & 0 & m & k-1 \end{array} \right\|$$

If $m \neq 0$: $\text{RANK}(A) = \text{RANK}(A|Y) = 3$: System has $\infty^{4-3} = \infty^1$ solutions, $\forall k \in \mathbb{R}$;

If $m=0$ and $k=1$: $\text{RANK}(A) = \text{RANK}(A|Y) = 2$: System has $\infty^{4-2} = \infty^2$ solutions.

If $m=0$ and $k \neq 1$: $\text{RANK}(A) = 2 \neq \text{RANK}(A|Y) = 3$:
 System has no solutions.

$$\text{II M1)} \quad g: \mathbb{R} \rightarrow \mathbb{R}^2; t \rightarrow (x_1; x_2); f: \mathbb{R}^2 \rightarrow \mathbb{R}^3; (x_1; x_2) \rightarrow (y_1; y_2; y_3)$$

MFEA3

Applying chain rule: $\frac{\partial(y_1; y_2; y_3)}{\partial(t)} = \frac{\partial(y_1; y_2; y_3)}{\partial(x_1; x_2)} \cdot \frac{\partial(x_1; x_2)}{\partial(t)}$.

$$y_f: \begin{cases} x_1 = \sin t \\ x_2 = \cos t \end{cases}; \begin{cases} y_1 = x_1 \cdot x_2 \\ y_2 = x_1 + x_2 \\ y_3 = x_1 - x_2 \end{cases} \quad \text{we get:}$$

$$\begin{aligned} \left\| \frac{dy_i}{dt} \right\| &= \left\| \begin{matrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} \end{matrix} \right\| \cdot \left\| \frac{dx_i}{dt} \right\| = \left\| \begin{matrix} x_2 & x_1 \\ 1 & 1 \\ 1 & -1 \end{matrix} \right\| \cdot \left\| \begin{matrix} \cos t \\ -\sin t \end{matrix} \right\| = \\ &= \left\| \begin{matrix} \cos t & \sin t \\ 1 & 1 \\ 1 & -1 \end{matrix} \right\| \cdot \left\| \begin{matrix} \cos t \\ -\sin t \end{matrix} \right\| = \left\| \begin{matrix} \cos^2 t - \sin^2 t \\ \cos t + \sin t \\ \cos t - \sin t \end{matrix} \right\|. \end{aligned}$$

$$\text{II M2)} \quad \begin{cases} f(x; y; z) = x \log y - y \log z + x y^3 z^2 = 0 \\ g(x; y; z) = x^3 y^2 + y^3 z^2 = 1 \end{cases} \quad \begin{cases} f(0; 1; 1) = 0 \\ g(0; 1; 1) = 1 \end{cases}$$

$$\frac{\partial(f; g)}{\partial(x; y; z)} = \left\| \begin{matrix} \log y + y^3 z^2 & \frac{x}{y} - \log z + 3x y^2 z^2 & -\frac{y}{z} + 2x y^3 z \\ 3x^2 y^2 & 2x^3 y + 3y^2 z^2 & 2y^3 z \end{matrix} \right\|$$

$$\frac{\partial(f; g)}{\partial(x; y; z)}(0; 1; 1) = \left\| \begin{matrix} 1 & 0 & -1 \\ 0 & 3 & 2 \end{matrix} \right\| \cdot \begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 3 \neq 0: \text{exists: } x \rightarrow (y(x); z(x)).$$

$$\text{so } \frac{dy}{dx}(0) = -\frac{\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix}} = -\frac{2}{3}; \quad \frac{dz}{dx}(0) = -\frac{\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix}} = -\frac{-3}{3} = 1.$$

Tangent vector equation at $x = 0$: $X \rightarrow (1; 1) + x(-\frac{2}{3}; 1)$:
 $x \rightarrow (1 - \frac{2}{3}x; 1 + x).$

II M3) $f(x; y; z) = x^2 - xz + yz^2$; f is twice differentiable as f is a polynomial. So $\mathcal{D}_v f = \nabla f \cdot v$ and $\mathcal{D}_{y,v}^2 f = v \cdot H(f) \cdot v^T$.

$$\nabla f = (2x - z; z^2; -x + 2yz); \quad \nabla f(1; 0; 1) = (1; 1; -1).$$

$$v = (\frac{1}{\sqrt{3}}; -\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}) \Rightarrow \mathcal{D}_v f(1; 0; 1) = (1; 1; -1) \cdot (\frac{1}{\sqrt{3}}; -\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

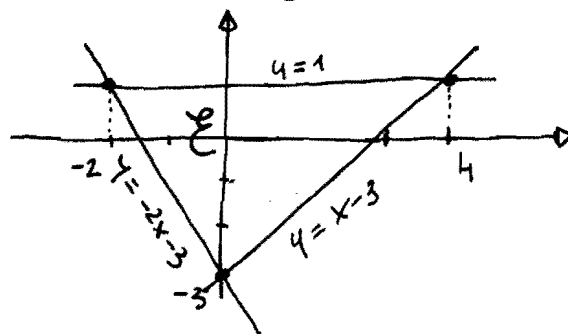
$$H(f) = \left\| \begin{matrix} 2 & 0 & -1 \\ 0 & 0 & 2z \\ -1 & 2z & 2y \end{matrix} \right\|; \quad H(f(1; 0; 1)) = \left\| \begin{matrix} 2 & 0 & -1 \\ 0 & 0 & 2 \\ -1 & 2 & 0 \end{matrix} \right\|.$$

$$\text{So } D_{y,v}^2 f(1;0;1) = \left\| \frac{1}{\sqrt{3}} \quad -\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \right\| \cdot \left\| \begin{matrix} 2 & 0 & -1 \\ 0 & 0 & 2 \\ -1 & 2 & 0 \end{matrix} \right\| \cdot \left\| \begin{matrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{matrix} \right\| =$$

$$= \left\| \frac{1}{\sqrt{3}} \quad \frac{2}{\sqrt{3}} \quad -\frac{3}{\sqrt{3}} \right\| \cdot \left\| \frac{1}{\sqrt{3}} \quad -\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \right\| = \frac{1}{3} - \frac{2}{3} - \frac{3}{3} = -\frac{4}{3}.$$

MPEA4

II M4) $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 - 3x + 2y \\ \text{s.t. } \begin{cases} 2x + y + 3 \geq 0 : g_1(x,y) \\ y \leq 1 : g_2(x,y) \\ y \geq x - 3 : g_3(x,y) \end{cases} \end{cases}$



$$\begin{cases} 2x + y = -3 \\ y = 1 \end{cases} \Rightarrow \begin{cases} 2x = -4 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = 1 \end{cases}$$

$$\begin{cases} 2x + y = -3 \\ y = x - 3 \end{cases} \Rightarrow \begin{cases} 3x = 0 \\ y = x - 3 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -3 \end{cases}$$

$$\begin{cases} y = x - 3 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = 4 \\ y = 1 \end{cases}. \text{ We do not use Kuhn-Tucker conditions.}$$

Free Max/min: $\begin{cases} f'_x = 2x - 3 = 0 \\ f'_y = 2y + 2 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2} \\ y = -1 \end{cases}. H(x,y) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = H(\frac{3}{2}; -1).$

So $\begin{cases} 2 > 0 \\ 2 \cdot 2 - 0 > 0 \end{cases} : (\frac{3}{2}; -1)$ is a minimum point.

We study f on $g_1: y = -2x - 3: f(x; -2x - 3) = x^2 + 4x^2 + 9 + 12x - 3x - 4x - 6 \Rightarrow$
 $\Rightarrow f(x) = 5x^2 + 5x + 3; f'(x) = 10x + 5 \geq 0 \Rightarrow x \geq -\frac{1}{2}$ $(x,y) = (-\frac{1}{2}; -2)$
 $-\frac{1}{2} : \text{minimum?}$

We study f on $g_2: y = 1: f(x; 1) = x^2 + 1 - 3x + 2 = x^2 - 3x + 3;$
 $\Rightarrow f'(x) = 2x - 3 \geq 0 \Rightarrow x \geq \frac{3}{2}$ $(x,y) = (\frac{3}{2}; 1)$
 $\frac{3}{2} : \text{minimum?}$

We study f on $g_3: y = x - 3: f(x; x - 3) = x^2 + x^2 + 9 - 6x - 3x + 2x - 6 = 2x^2 - 7x + 3;$
 $\Rightarrow f'(x) = 4x - 7 \geq 0 \Rightarrow x \geq \frac{7}{4}$ $(x,y) = (\frac{7}{4}; -\frac{5}{4}).$
 $\frac{7}{4} : \text{minimum?}$

We use Lagrangian function on $g_1: x^2 + y^2 - 3x + 2y - \lambda(-2x - y - 3)$

$$\begin{cases} \Lambda'_x = 2x - 3 + 2\lambda = 0 \\ \Lambda'_y = 2y + 2 + \lambda = 0 \\ 2x + y + 3 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2} - \lambda \\ y = -1 - \frac{1}{2}\lambda \\ 3 - 2\lambda - 1 - \frac{1}{2}\lambda + 3 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2} - \lambda \\ y = -1 - \frac{1}{2}\lambda \\ \frac{5}{2}\lambda = 5 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} \\ y = -2 \\ \lambda = 2 \end{cases} \text{ (Max?)}$$

$\bar{H} = \begin{vmatrix} 0 & -2 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} \Rightarrow |\bar{H}| = (-1)(2) + 2(-4) < 0 : \text{minimum on the constraint,}$
 but $\lambda = 2 > 0$, so this point is not a maximum nor a minimum point.

We use the Lagrangian function on $g_2: x^2 + y^2 - 3x + 2y - \lambda(y-1)$

MPEA5

$$\begin{cases} \lambda'_x = 2x - 3 = 0 \\ \lambda'_y = 2y + 2 - \lambda = 0 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2} \\ y = 1 \\ \lambda = 4 \end{cases} \text{ (Max?)}$$

$$\bar{H} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} \Rightarrow |\bar{H}| = 1 \cdot (-2) < 0 : \text{minimum on the constraint, but}$$

$\lambda = 4 > 0$, so this point is not a maximum nor a minimum point.

We use the Lagrangian function on $g_3: x^2 + y^2 - 3x + 2y - \lambda(x - y - 3)$.

$$\begin{cases} \lambda'_x = 2x - 3 - \lambda = 0 \\ \lambda'_y = 2y + 2 + \lambda = 0 \\ y = x - 3 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2} + \frac{\lambda}{2} \\ y = -1 - \frac{1}{2}\lambda \\ -1 - \frac{1}{2}\lambda = \frac{3}{2} + \frac{1}{2}\lambda - 3 \end{cases} \Rightarrow \begin{cases} x = \frac{7}{4} \\ y = -\frac{5}{4} \\ \lambda = \frac{1}{2} \end{cases} \text{ (Max?)}$$

$$\bar{H} = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} \Rightarrow |\bar{H}| = (-1)(2) + 2(-1) < 0 : \text{minimum on the constraint,}$$

but $\lambda = \frac{1}{2} > 0$, so this point is not a maximum nor a minimum point.

So we have:

a minimum (free) point: Absolute.

at $(\frac{3}{2}; -1)$; $f(\frac{3}{2}; -1) = -\frac{13}{4}$.

Then we have Maximum points at:

$(-2; 1)$; $f(-2; 1) = 13$: Absolute

$(0; -3)$; $f(0; -3) = 3$: Relative (local)

$(4; 1)$; $f(4; 1) = 7$: Relative (local)

