Task Mathematics for Economic Applications. 24/3/2012 $\boxed{\text{MfEA 1}}$

IM1) $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\mathrm{sen}\,\frac{\pi}{4}\right)$

$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{7}{4}\pi + i\,\mathrm{sen}\,\frac{7}{4}\pi\right).$

$(1+i)^8 - (1-i)^8 = (\sqrt{2})^8\left(\cos 8\cdot\frac{\pi}{4} + i\,\mathrm{sen}\,8\cdot\frac{\pi}{4}\right) - (\sqrt{2})^8\left(\cos 8\cdot\frac{7}{4}\pi + i\,\mathrm{sen}\,8\cdot\frac{7}{4}\pi\right) =$

$= 2^4(\cos 2\pi + i\,\mathrm{sen}\,2\pi) - 2^4(\cos 14\pi + i\,\mathrm{sen}\,14\pi) = 16 - 16 = 0.$

IM2) $(A|Y) = \left\|\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 2 \\ 1 & -1 & 2 & 1 & 1 \\ 1 & -3 & m & K & -1 \end{array}\right\| \to \left\|\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 2 & -1 \\ 0 & -3 & m-1 & K+1 & -3 \end{array}\right\| \to \left\|\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & m-4 & K-5 & 0 \end{array}\right\|$

If $m \neq 4$ or $K \neq 5$: $RANK(A) = RANK(A|Y) = 3$: System has ∞^1 solutions.

If $m = 4$ and $K = 5$: $RANK(A) = RANK(A|Y) = 2$: System has ∞^2 solutions.

System is never impossible.

IM3) $A = \left\|\begin{matrix} 1 & 0 & K \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right\| \to \left|\begin{matrix} 1-\lambda & 0 & K \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{matrix}\right| = (1-\lambda)\left[(1-\lambda)^2 - 0\right] + 1\cdot\left(0 - K(1-\lambda)\right) =$

$= (1-\lambda)(1+\lambda^2-2\lambda-K) = (1-\lambda)(\lambda^2-2\lambda+1-K) = 0.$ $\lambda = 1$ eigenvalue for $A\ \forall K \in \mathbb{R}$.

$\lambda = 1$ is a multiple eigenvalue if : $1-2+1-K = -K = 0 \Rightarrow K = 0.$

$\lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$ for $K = 0$.

If $K=0$ and $\lambda=1$: $\|A - 1\cdot\mathbb{I}\| = \left\|\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}\right\| \Rightarrow RANK(A - 1\cdot\mathbb{I}) = 1 \Rightarrow m_1^g = 3-1 = 2.$

But $m_1^a = 3 > 2 = m_1^g$ so A is not diagonalizable for $K=0$.

A can have multiple eigenvalues if $\lambda^2 - 2\lambda + 1 - K$ has a double root.

But $\lambda^2 - 2\lambda + 1 - K = 0 \Rightarrow \lambda = 1 \pm \sqrt{1-1+K} = 1 \pm \sqrt{K}$ so $\Delta = 0$ iff $K = 0$

and this case has just been solved.

Then, $\lambda = 0$ is an eigenvalue for A iff $|A| = 0 \Rightarrow 1\cdot(1-K) = 0 \Rightarrow K = 1.$

If $K=1$: $\lambda^2 - 2\lambda + 1 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2) = 0 \Rightarrow \lambda_1 = 1;\ \lambda_2 = 0;\ \lambda_3 = 2.$

<u>If $K=1$ and $\lambda=1$</u> : $(A - 1\cdot\mathbb{I})\cdot X = \underline{0} \Rightarrow \left\|\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases} z = 0 \\ \forall y \\ x = 0 \end{cases} \Rightarrow (0;1;0)$ unit eigenvector.

<u>If $K=1$ and $\lambda=0$</u>: $(A - 0\mathbb{I})\cdot X = \underline{0} \Rightarrow \left\|\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases} x+z = 0 \\ y = 0 \\ x+z = 0 \end{cases} \Rightarrow \begin{cases} z = -x \\ y = 0 \end{cases} \Rightarrow \left(\frac{1}{\sqrt{2}}; 0; -\frac{1}{\sqrt{2}}\right)$ unit eigenvector

<u>If $K=1$ and $\lambda=2$</u>: $(A - 2\mathbb{I})\cdot X = \underline{0} \Rightarrow \left\|\begin{matrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases} -x+z = 0 \\ y = 0 \\ x-z = 0 \end{cases} \Rightarrow \begin{cases} z = x \\ y = 0 \end{cases} \Rightarrow \left(\frac{1}{\sqrt{2}}; 0; \frac{1}{\sqrt{2}}\right)$ unit eigenvector.

So $U = \left\|\begin{matrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{matrix}\right\|$ is an orthogonal matrix which diagonalizes A:

$$U^T \cdot A \cdot U = D = \left\|\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{matrix}\right\|.$$

I M4) $X = \begin{Vmatrix} 1 & 1 \\ 1 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} -1 \\ 2 \end{Vmatrix} = (-1)\cdot(1;1) + 2\cdot(1;2) = \begin{Vmatrix} 1 \\ 3 \end{Vmatrix}$.

$\boxed{\text{MfEA 2}}$

To get coordinates in the base $\{(2;1);(1;-1)\}$:

$$\begin{Vmatrix} 1 \\ 3 \end{Vmatrix} = \begin{Vmatrix} 2 & 1 \\ 1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} \Rightarrow \begin{cases} 2x_1 + x_2 = 1 \\ x_1 - x_2 = 3 \end{cases} \Rightarrow \begin{cases} 2x_2 + 6 + x_2 = 1 \\ x_1 = x_2 + 3 \end{cases} \Rightarrow \begin{cases} 3x_2 = -5 \\ x_1 = x_2 + 3 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_2 = -\frac{5}{3} \\ x_1 = -\frac{5}{3} + 3 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{4}{3} \\ x_2 = -\frac{5}{3} \end{cases}$$

. With another method: $\begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} = \begin{Vmatrix} 2 & 1 \\ 1 & -1 \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} 1 \\ 3 \end{Vmatrix}$.

II M1) $f(x;y) = xy\cdot(x^2+y^2-1) = x^3y + xy^3 - xy$.

$$\begin{cases} f'_x = 3x^2y + y^3 - y = y(3x^2+y^2-1) = 0 \\ f'_y = x^3 + 3xy^2 - x = x(x^2+3y^2-1) = 0 \end{cases} \Rightarrow \begin{cases} y=0 \\ x=0 \end{cases} \cup \begin{cases} y=0 \\ x^2+3y^2-1=0 \end{cases} \cup \begin{cases} 3x^2+y^2-1=0 \\ x=0 \end{cases} \cup$$

$$\cup \begin{cases} 3x^2+y^2-1=0 \\ x^2+3y^2-1=0 \end{cases}$$

. I) $\begin{cases} x=0 \\ y=0 \end{cases}$; II) $\begin{cases} y=0 \\ x^2+3y^2-1=0 \end{cases} \Rightarrow \begin{cases} y=0 \\ x^2=1 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=0 \end{cases}$ and $\begin{cases} x=-1 \\ y=0 \end{cases}$;

III $\begin{cases} x=0 \\ 3x^2+y^2-1=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y^2=1 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \end{cases}$ and $\begin{cases} x=0 \\ y=-1 \end{cases}$; IV $\begin{cases} 3x^2+y^2-1=0 \\ x^2+3y^2-1=0 \end{cases}$ $(1^{\circ}-2^{\circ}) \Rightarrow$

$$\Rightarrow \begin{cases} 2x^2-2y^2=0 \\ x^2+3y^2=1 \end{cases} \Rightarrow \begin{cases} y^2=x^2 \\ x^2+3y^2=1 \end{cases} \Rightarrow \begin{cases} y^2=x^2 \\ 4x^2=1 \end{cases} \Rightarrow \begin{cases} y^2=x^2 \\ x^2=\frac{1}{4} \end{cases} \Rightarrow \begin{cases} x=\frac{1}{2} \\ y^2=\frac{1}{4} \end{cases} \Rightarrow \begin{cases} x=\frac{1}{2} \\ y=\frac{1}{2} \end{cases} \text{ and } \begin{cases} x=\frac{1}{2} \\ y=-\frac{1}{2} \end{cases};$$

and $\begin{cases} x=-\frac{1}{2} \\ y^2=\frac{1}{4} \end{cases} \Rightarrow \begin{cases} x=-\frac{1}{2} \\ y=\frac{1}{2} \end{cases}$ and $\begin{cases} x=-\frac{1}{2} \\ y=\frac{1}{2} \end{cases}$. So we get 9 stationary points:

$(0;0)$; $(1;0)$; $(-1;0)$; $(0;1)$; $(0;-1)$; $(\frac{1}{2};\frac{1}{2})$; $(\frac{1}{2};-\frac{1}{2})$; $(-\frac{1}{2};\frac{1}{2})$; $(-\frac{1}{2};-\frac{1}{2})$.

$H = \begin{Vmatrix} 6xy & 3x^2+3y^2-1 \\ 3x^2+3y^2-1 & 6xy \end{Vmatrix}$. $H(0;0) = \begin{Vmatrix} 0 & -1 \\ -1 & 0 \end{Vmatrix}$: Saddle point;

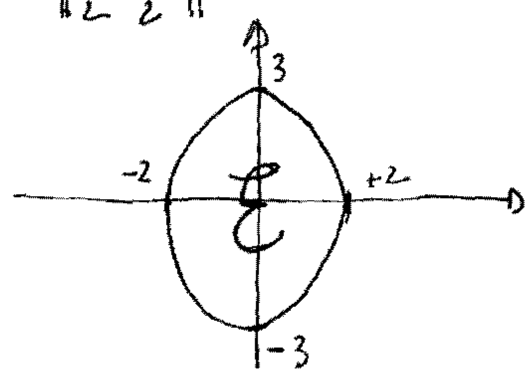
$H(1;0) = \begin{Vmatrix} 0 & 2 \\ 2 & 0 \end{Vmatrix}$: Saddle point; $H(-1;0) = \begin{Vmatrix} 0 & 2 \\ 2 & 0 \end{Vmatrix}$: Saddle point;

$H(0;1) = \begin{Vmatrix} 0 & 2 \\ 2 & 0 \end{Vmatrix}$: Saddle point; $H(0;-1) = \begin{Vmatrix} 0 & 2 \\ 2 & 0 \end{Vmatrix}$: Saddle point;

$H(\frac{1}{2};\frac{1}{2}) = \begin{Vmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{Vmatrix}$: Minimum point; $H(\frac{1}{2};-\frac{1}{2}) = \begin{Vmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{Vmatrix}$: Maximum point;

$H(-\frac{1}{2};\frac{1}{2}) = \begin{Vmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{Vmatrix}$: Maximum point; $H(-\frac{1}{2};-\frac{1}{2}) = \begin{Vmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{Vmatrix}$: Minimum point.

II M2) $\begin{cases} \text{Max/min } f(x;y) = x^2+y^2 \\ \text{s.t. } \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \end{cases}$



$\mathcal{E}$ is a bounded and closed set. $f(x;y)$ is a continuous function, so Weierstrass Theorem holds. Constraint is qualified.

$\Lambda = x^2+y^2-\lambda\left(\frac{x^2}{4}+\frac{y^2}{9}-1\right)$ $\boxed{\text{MfEA3}}$

<u>for $\lambda = 0$</u> $\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 2y = 0 \\ \frac{x^2}{4}+\frac{y^2}{9} \leq 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0+0 \leq 1 \end{cases}$; $H = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix} = H(0;0)$: Minimum point.

satisfied

<u>for $\lambda \neq 0$</u>

$$\begin{cases} \Lambda'_x = 2x - \frac{1}{2}\lambda x = x\left(2-\frac{1}{2}\lambda\right) = 0 \\ \Lambda'_y = 2y - \frac{2}{9}\lambda y = y\left(2-\frac{2}{9}\lambda\right) = 0 \\ \frac{x^2}{4}+\frac{y^2}{9} = 1 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0+0=1 \end{cases} \cup \begin{cases} x=0 \\ \lambda=9 \\ y^2=9 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=3 \\ \lambda=9 \end{cases} \text{ and } \begin{cases} x=0 \\ y=-3 \\ \lambda=9 \end{cases} \cup$$

No solution; Max?; Max?

$$\cup \begin{cases} \lambda=4 \\ y=0 \\ x^2=4 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=0 \\ \lambda=4 \end{cases} \text{ and } \begin{cases} x=-2 \\ y=0 \\ \lambda=4 \end{cases} \cup \begin{cases} \lambda=4 \\ \lambda=9 \\ \text{impossible} \end{cases}. \quad \overline{H} = \begin{Vmatrix} 0 & \frac{x}{2} & \frac{2}{9}y \\ \frac{x}{2} & 2-\frac{\lambda}{2} & 0 \\ \frac{2}{9}y & 0 & 2-\frac{2}{9}\lambda \end{Vmatrix}.$$

Max?; Max?

$$\overline{H}(0;3;9) = \begin{Vmatrix} 0 & 0 & \frac{2}{3} \\ 0 & -\frac{5}{2} & 0 \\ \frac{2}{3} & 0 & 0 \end{Vmatrix} \Rightarrow \frac{2}{3}\left(0+\frac{5}{3}\right) > 0: \ (0;3) \text{ is a Maximum point.}$$

$$\overline{H}(0;-3;9) = \begin{Vmatrix} 0 & 0 & -\frac{2}{3} \\ 0 & -\frac{5}{2} & 0 \\ -\frac{2}{3} & 0 & 0 \end{Vmatrix} \Rightarrow -\frac{2}{3}\cdot\left(0-\frac{5}{3}\right) > 0: \ (0;-3) \text{ is a Maximum point.}$$

$$\overline{H}(2;0;4) = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{10}{9} \end{Vmatrix} \Rightarrow -1\cdot\left(\frac{10}{9}-0\right) < 0: \text{ Min? so } (2;0) \text{ is not a Maximum nor a minimum point.}$$

$$\overline{H}(-2;0;4) = \begin{Vmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{10}{9} \end{Vmatrix} \Rightarrow 1\cdot\left(-\frac{10}{9}-0\right) < 0: \text{ Min? so } (-2;0) \text{ is not a Maximum nor a minimum point.}$$

II M3) $f(x;y) = x^3y - 2x^2y^2 + xy^3 + 4 = 0$; $f(1;-1) = -1-2-1+4 = 0$ satisfied.

$\nabla f = (3x^2y - 4xy^2 + y^3;\ x^3 - 4x^2y + 3xy^2)$; $\nabla f(1;-1) = (-3-4-1;\ 1+4+3) = (-8;8)$.

So $f'_y \neq 0$: exists an implicit function $x \to y(x)$.

$$H(f) = \begin{Vmatrix} 6xy-4y^2 & 3x^2-8xy+3y^2 \\ 3x^2-8xy+3y^2 & 6xy-4x^2 \end{Vmatrix}; \ H(1;-1) = \begin{Vmatrix} -10 & 14 \\ 14 & -10 \end{Vmatrix}.$$

$$y'(1) = -\frac{-8}{8} = 1; \ y''(1) = -\frac{-10+2\cdot14\cdot1+(-10)\cdot1}{8} = -\frac{-10+28-10}{8} = -1.$$

MfEA4

$P_2(x;1) = y(1) + y'(1)(x-1) + \frac{1}{2} y''(1) \cdot (x-1)^2$

$P_2(x;1) = -1 + 1(x-1) + \frac{1}{2} \cdot (-1) \cdot (x-1)^2.$

$P_2(x;1) = -1 + x - 1 - \frac{1}{2}(x^2 - 2x + 1) = x - 2 - \frac{1}{2}x^2 + x - \frac{1}{2} = -\frac{1}{2}x^2 + 2x - \frac{5}{2}.$

II M4) $f(x;y) = x^2 - xy + y^2$: differentiable function.

$u = \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$; $v = \left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right)$; $\nabla f(x;y) = (2x - y; 2y - x).$

$$\begin{cases} D_u f(x_0;y_0) = (2x_0 - y_0; 2y_0 - x_0) \cdot \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(2x_0 - y_0 + 2y_0 - x_0) = \sqrt{2} \\ D_v f(x_0;y_0) = (2x_0 - y_0; 2y_0 - x_0) \cdot \left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(2x_0 - y_0 - 2y_0 + x_0) = 3\sqrt{2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_0 + y_0 = 2 \\ 3x_0 - 3y_0 = 6 \end{cases} \Rightarrow \begin{cases} x_0 + y_0 = 2 \\ x_0 - y_0 = 2 \end{cases} \Rightarrow \begin{cases} x_0 = 2 \\ y_0 = 0 \end{cases}.$$

The point $(x_0;y_0)$ is $(2;0)$.