Mathematics for Economic Applications Intermediate test 12/12/2012

MAEF1

IM1) $\frac{i^8}{(1+i)^6} - (1-i)^4$. $\quad i^8 = (i^2)^4 = (-1)^4 = 1$.

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\text{sen}\,\frac{\pi}{4}\right)$; $(1+i)^6 = (\sqrt{2})^6\left(\cos\frac{6\cdot\pi}{4} + i\,\text{sen}\,\frac{6\cdot\pi}{4}\right) = 8\left(\cos\frac{3}{2}\pi + i\,\text{sen}\,\frac{3}{2}\pi\right) = -8i$.

$1-i = \sqrt{2}\left(\cos\frac{7}{4}\pi + i\,\text{sen}\,\frac{7}{4}\pi\right)$; $(1-i)^4 = (\sqrt{2})^4\cdot\left(\cos 4\cdot\frac{7}{4}\pi + i\,\text{sen}\,4\cdot\frac{7}{4}\pi\right) = 4(\cos\pi + i\,\text{sen}\,\pi) = -4$.

$\frac{i^8}{(1+i)^6} - (1-i)^4 = \frac{1}{-8i} - (-4) = \frac{1}{8}i + 4$.

IM2) $A = \left\|\begin{matrix} 1 & 3 & 4 \\ 2 & 0 & -4 \\ -1 & -3 & -4 \end{matrix}\right\| \to \begin{vmatrix} 1-\lambda & 3 & 4 \\ 2 & -\lambda & -4 \\ -1 & -3 & -4-\lambda \end{vmatrix} \underset{R_1+R_3}{=} \begin{vmatrix} -\lambda & 0 & -\lambda \\ 2 & -\lambda & -4 \\ -1 & -3 & -4-\lambda \end{vmatrix} = (-\lambda)\big((-\lambda)(-4-\lambda) - 12\big) - \lambda(-6-\lambda) =$

$= (-\lambda)(\lambda^2 + 4\lambda - 12 - 6 - \lambda) = (-\lambda)(\lambda^2 + 3\lambda - 18) = 0 \Rightarrow (-\lambda)(\lambda+6)(\lambda-3) = 0 \Rightarrow$

Eigenvalues: $\lambda_1 = 0$; $\lambda_2 = -6$; $\lambda_3 = 3$.

$B = \left\|\begin{matrix} -2 & 2 & -2 \\ -2 & 2 & 1 \\ -6 & 6 & -3 \end{matrix}\right\| \to \begin{vmatrix} -2-\lambda & 2 & -2 \\ -2 & 2-\lambda & 1 \\ -6 & 6 & -3-\lambda \end{vmatrix} \underset{C_1+C_2}{=} \begin{vmatrix} -\lambda & 2 & -2 \\ -\lambda & 2-\lambda & 1 \\ 0 & 6 & -3-\lambda \end{vmatrix} = (-\lambda)\big((2-\lambda)(-3-\lambda) - 6\big) + \lambda(-6 - 2\lambda + 12) =$

$= (-\lambda)(\lambda^2 + \lambda - 12 + 2\lambda - 6) = (-\lambda)(\lambda^2 + 3\lambda - 18) = 0 \Rightarrow (-\lambda)(\lambda+6)(\lambda-3) = 0 \Rightarrow$

Eigenvalues: $\lambda_1 = 0$; $\lambda_2 = -6$; $\lambda_3 = 0$.

The matrices have the same eigenvalues. The eigenvalues are distinct, so the matrices are diagonalizable, and they are similar to the same diagonal matrix, so they are similar matrices.

IM3) $\left\|\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & K \end{matrix}\right\| \to \left\|\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & K \end{matrix}\right\| \to \left\|\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & K-1 \end{matrix}\right\|$.

As $\text{Dim}(\text{Ker}) = 2 \Rightarrow 2 = 4 - \text{RANK}(A) \Rightarrow \text{RAN}(A) = 2 \Rightarrow K - 1 = 0 \Rightarrow K = 1$.

To find a basis for the Image:

$\left\|\begin{matrix} 1 & 1 & 1 & 0 & \vdots & y_1 \\ 0 & 1 & 1 & 1 & \vdots & y_2 \\ 1 & 2 & 2 & 1 & \vdots & y_3 \end{matrix}\right\| \to \left\|\begin{matrix} 1 & 1 & 1 & 0 & \vdots & y_1 \\ 0 & 1 & 1 & 1 & \vdots & y_2 \\ 0 & 1 & 1 & 1 & \vdots & y_3 - y_1 \end{matrix}\right\| \to \left\|\begin{matrix} 1 & 1 & 1 & 0 & \vdots & y_1 \\ 0 & 1 & 1 & 1 & \vdots & y_2 \\ 0 & 0 & 0 & 0 & \vdots & y_3 - y_1 - y_2 \end{matrix}\right\| \Rightarrow$

$\Rightarrow y_3 - y_1 - y_2 = 0 \Rightarrow y_3 = y_1 + y_2 \quad : (y_1; y_2; y_1 + y_2) \Rightarrow$

Basis: $\{(1;0;1);(0;1;1)\}$.

To find a basis for the Kernel:

$\left\|\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{matrix}\right\| \left\|\begin{matrix} x \\ y \\ z \\ w \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \left\|\begin{matrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \\ w \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} x + y + z = 0 \\ y + z + w = 0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} z = -x-y \\ w = -y-z = -y+x+y = x \end{cases} \Rightarrow \begin{cases} z = -x-y \\ w = x. \end{cases} \Rightarrow (x; y; -x-y; x).$$

$\boxed{\text{MAEF2}}$

Basis: $\{(1;0;-1;1);(0;1;-1;0)\}$.

I M4) $$\begin{cases} x-y+2z=-1 \\ 2x+y+z=1 \\ -2x+my+kz=1 \end{cases} \Rightarrow \left\| \begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 2 & 1 & 1 & 1 \\ -2 & m & k & 1 \end{array} \right\| \Rightarrow \left\| \begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & m+1 & k+1 & 2 \end{array} \right\|.$$

The system has ∞^1 solutions iff $\text{Rank}(A) = \text{Rank}(A|Y) = 2 \Rightarrow$

$$\Rightarrow \frac{m+1}{3} = \frac{k+1}{-3} = \frac{2}{3} \Rightarrow \begin{cases} m+1=2 \\ k+1=-2 \end{cases} \Rightarrow \begin{cases} m=1 \\ k=-3 \end{cases} \Rightarrow \left\| \begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & 2 & -2 & 2 \end{array} \right\| \Rightarrow$$

$$\Rightarrow \left\| \begin{matrix} 1 & -1 & 2 \\ 0 & 3 & -3 \end{matrix} \right\| \cdot \left\| \begin{matrix} x \\ y \\ z \end{matrix} \right\| = \left\| \begin{matrix} -1 \\ 3 \end{matrix} \right\| \Rightarrow \begin{cases} x-y+2z=-1 \\ 3y-3z=3 \end{cases} \Rightarrow \begin{cases} x = y-2z-1 = z+1-2z-1 = -z \\ y = z+1 \end{cases}.$$

Solutions: $(-z; z+1; z)$.

I M5) $$\left\| \begin{matrix} 1 & 0 & 0 \\ 0 & k & 1 \\ 0 & 1 & 1 \end{matrix} \right\| \to \left| \begin{matrix} 1-\lambda & 0 & 0 \\ 0 & k-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{matrix} \right| = (1-\lambda)\cdot\left[(k-\lambda)(1-\lambda)-1\right] = (1-\lambda)(\lambda^2-\lambda-k\lambda+k-1) =$$

$$= (1-\lambda)(\lambda^2-(k+1)\lambda+k-1) = 0.$$

We have multiple eigenvalues if:

•) $\lambda^2-(k+1)\lambda+k-1=0$ for $\lambda=1 \Rightarrow 1-k-1+k-1=-1=0$: impossible.

••) $\lambda^2-(k+1)\lambda+k-1$ has two real not distinct roots, i.e. iff

$\Delta = (k+1)^2-4(k-1)=0 \Rightarrow k^2+2k+1-4k+4 = k^2-2k+5=0 \Rightarrow$

$k = 1 \pm \sqrt{1-5}$: there are no real solutions.

So the matrix never has multiple eigenvalue, $\forall k \in \mathbb{R}$.