Task Mathematics for Economic Applications 29/1/2013 $\boxed{\text{MfEA1}}$

IM1) $z^4-2z^3-2z^2+8z-8=0$ if $z=2$: $16-16-8+16-8=0$.

$$\begin{array}{c|cccc|c} & 1 & -2 & -2 & 8 & -8 \\ 2 & & 2 & 0 & -4 & 8 \\ \hline & 1 & 0 & -2 & 4 & 0 \end{array} \Rightarrow (z-2)(z^3-2z+4)=0 : \text{ if } z=-2: -8+4+4=0.$$

$$\begin{array}{c|ccc|c} & 1 & 0 & -2 & 4 \\ -2 & & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & 0 \end{array} \Rightarrow (z-2)(z+2)(z^2-2z+2)=0 \Rightarrow z=1\pm\sqrt{1-2}=1\pm i.$$

So: $z_1\cdot z_2\cdot z_3\cdot z_4 = 2\cdot(-2)\cdot(1+i)(1-i) = -4\cdot(1+1) = -8.$

$-8 = 8(\cos\pi + i\,\mathrm{sen}\,\pi) \Rightarrow \sqrt[3]{-8} = \sqrt[3]{8}\left(\cos\left(\frac{\pi}{3}+k\cdot\frac{2\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{3}+k\frac{2\pi}{3}\right)\right): 0\le k\le 2.$

$k=0$: $2\cdot\left(\cos\frac{\pi}{3}+i\,\mathrm{sen}\,\frac{\pi}{3}\right) = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 1+\sqrt{3}\cdot i$;

$k=1$: $2\cdot(\cos\pi + i\,\mathrm{sen}\,\pi) = -2$;

$k=2$: $2\cdot\left(\cos\frac{5}{3}\pi+i\,\mathrm{sen}\,\frac{5}{3}\pi\right) = 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 1-\sqrt{3}\cdot i.$

IM2) A similar to B : $A\cdot P = P\cdot B \Rightarrow B = P^{-1}\cdot A\cdot P$; $P=\begin{Vmatrix}1&0&1\\0&-1&1\\0&0&1\end{Vmatrix} \Rightarrow |P|=-1.$

$$\left\|\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&-1&1&0&1&0\\0&0&1&0&0&1\end{array}\right\| \xrightarrow[R_2\leftarrow(-1)\cdot R_2]{} \left\|\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&-1&0&-1&0\\0&0&1&0&0&1\end{array}\right\| \xrightarrow[R_2\leftarrow R_2+R_3]{} \left\|\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&0&0&-1&1\\0&0&1&0&0&1\end{array}\right\| \xrightarrow[R_1\leftarrow R_1-R_3]{} \left\|\begin{array}{ccc|ccc}1&0&0&1&0&-1\\0&1&0&0&-1&1\\0&0&1&0&0&1\end{array}\right\| : P^{-1}=\begin{Vmatrix}1&0&-1\\0&-1&1\\0&0&1\end{Vmatrix}.$$

$$B=P^{-1}\cdot A\cdot P=\begin{Vmatrix}1&0&-1\\0&-1&1\\0&0&1\end{Vmatrix}\cdot\begin{Vmatrix}1&2&1\\2&-1&-1\\1&0&1\end{Vmatrix}\cdot\begin{Vmatrix}1&0&1\\0&-1&1\\0&0&1\end{Vmatrix}=\begin{Vmatrix}0&2&0\\-1&1&2\\1&0&1\end{Vmatrix}\cdot\begin{Vmatrix}1&0&1\\0&-1&1\\0&0&1\end{Vmatrix}=\begin{Vmatrix}0&-2&2\\-1&-1&2\\1&0&2\end{Vmatrix}=B.$$

As matrices are similar matrices, they have the same eigenvalues. We can use, to find eigenvalues, A or B. We use B.

$$\begin{vmatrix}-\lambda&-2&2\\-1&-1-\lambda&2\\1&0&2-\lambda\end{vmatrix} \underset{C_2\leftarrow C_2+C_3}{=} \begin{vmatrix}-\lambda&0&2\\-1&1-\lambda&2\\1&2-\lambda&2-\lambda\end{vmatrix} = (-\lambda)\Big((1-\lambda)(2-\lambda)-2(2-\lambda)\Big)+2\Big(-1\cdot(2-\lambda)-1(1-\lambda)\Big) =$$

$$= (-\lambda)(\lambda^2-3\lambda+2-4+2\lambda)+2(\lambda-2-1+\lambda) = (-\lambda)(\lambda^2-\lambda-2)+4\lambda-6 =$$

$= -\lambda^3+\lambda^2+2\lambda+4\lambda-6=0 \Rightarrow \lambda^3-\lambda^2-6\lambda+6=0$. If $\lambda=1$: $1-1-6+6=0 \Rightarrow$

$$\begin{array}{c|ccc|c} & 1 & -1 & -6 & 6 \\ 1 & & 1 & 0 & -6 \\ \hline & 1 & 0 & -6 & 0 \end{array} \Rightarrow (\lambda-1)(\lambda^2-6)=0 \Rightarrow \lambda_1=1;\ \lambda_2=\sqrt{6};\ \lambda_3=-\sqrt{6}.$$

Eigenvalues are distinct, so B (and A) are diagonalisable matrices.

IM3) $\begin{Vmatrix}3&-1&1\\0&2&0\\m&k&3\end{Vmatrix} \to \begin{vmatrix}3-\lambda&-1&1\\0&2-\lambda&0\\m&k&3-\lambda\end{vmatrix} = (2-\lambda)\big((3-\lambda)(3-\lambda)-m\big) = (2-\lambda)(\lambda^2-6\lambda+9-m) = 0.$

•) If $\lambda=2$: $\lambda^2-6\lambda+9-m = 4-12+9-m = 1-m = 0$ if $m=1$.

If $m=1$: $\lambda^2-6\lambda+8 = (\lambda-2)(\lambda-4) \Rightarrow \lambda_1=\lambda_2=2$; $\lambda_3=4$.

$\|A-2\mathbb{I}\| = \begin{Vmatrix}1&-1&1\\0&0&0\\1&k&1\end{Vmatrix}$: If $k=-1 \Rightarrow \mathrm{RANK}(A-2\mathbb{I})=1 \Rightarrow m_2^g = 3-1 = 2 = m_2^a \Rightarrow$

$\Rightarrow$ if $m=1$ and $k=-1 \Rightarrow A$ is a diagonalisable matrix having a multiple eigenvalue : $\lambda=2$. $\boxed{MfEA2}$

$\cdot\cdot)$ $\lambda^2-6\lambda+9-m=0 \Rightarrow \lambda = 3\pm\sqrt{9-9+m} = 3\pm\sqrt{m}$. If $m=0$: $\lambda_1=2$; $\lambda_2=\lambda_3=3$.

$$\|A-3\mathbb{I}\| = \begin{Vmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & K & 0 \end{Vmatrix}; \text{ since } \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 1 \neq 0 \ \text{RANK}(A-3\mathbb{I}) = 2 \ \forall K\in\mathbb{R}.$$

And so, if $m=0$ the matrix is not diagonalisable.

I M4)
$$\begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ K & 0 & 0 & 1 \end{Vmatrix} \xrightarrow[R_4\leftarrow R_4-K\cdot R_1]{} \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -K & 0 & 1 \end{Vmatrix} \xrightarrow[R_4\leftarrow R_4+KR_2]{} \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & K & 1 \end{Vmatrix} \xrightarrow[R_4\leftarrow R_4-KR_3]{} \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1-K \end{Vmatrix}$$

$\Rightarrow \text{RANK}(A) = 3$ iff $K=1$.

$\Rightarrow \text{Dim}(\text{Ker}(A)) = 4-3 = 1$: MAX.

$$f(1;-1;1;-1) = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}.$$

II M1)
$$\begin{cases} f(1;y;z) = z-2y=0 \\ g(1;y;z) = 1+y^2-4y+z=0 \end{cases} \Rightarrow \begin{cases} z=2y \\ y^2-2y+1=0 \end{cases} \Rightarrow \begin{cases} z=2y \\ (y-1)^2=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \\ z=2 \end{cases}.$$

$$\frac{\partial(f;g)}{\partial(x;y;z)} = \begin{Vmatrix} -2y & -2x & 1 \\ 2x-8xy & 2y-4x^2 & 1 \end{Vmatrix}; \quad \frac{\partial(f;g)}{\partial(x;y;z)}(1;1;2) = \begin{Vmatrix} -2 & -2 & 1 \\ -6 & -2 & 1 \end{Vmatrix}.$$

It is not possible to define : $x \to (y(x);z(x))$ since $\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix} = 0$.

We define : $z \to (x(z);y(z))$ and so:

$$\frac{dx}{dz} = -\frac{\begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{0}{-8} = 0; \quad \frac{dy}{dz} = -\frac{\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix}}{\begin{vmatrix} -2 & -2 \\ -6 & -2 \end{vmatrix}} = -\frac{4}{-8} = \frac{1}{2}.$$

We can also define : $y \to (x(y);z(y))$ and so:

$$\frac{dx}{dy} = -\frac{\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix}} = -\frac{0}{4} = 0; \quad \frac{dz}{dy} = -\frac{\begin{vmatrix} -2 & -2 \\ -6 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix}} = -\frac{-8}{4} = 2.$$

II M2) $f(x;y) = xy(1-x^2-y) = xy - x^3y - xy^2$.

$$\nabla f = \underline{0} \Rightarrow \begin{cases} f'_x = y-3x^2y-y^2 = y(1-3x^2-y) = 0 \\ f'_y = x-x^3-2xy = x(1-x^2-2y) = 0 \end{cases} \Rightarrow \begin{cases} y=0 \\ x=0 \end{cases} \cup \begin{cases} y=0 \\ 1-x^2=0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x=1 \\ y=0 \end{cases} e \begin{cases} x=-1 \\ y=0 \end{cases} \cup \begin{cases} x=0 \\ 1-y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \end{cases} \cup \begin{cases} 1-3x^2-y=0 \\ 1-x^2-2y=0 \end{cases} \Rightarrow \begin{cases} y=1-3x^2 \\ 1-x^2-2+6x^2=0 \end{cases} \Rightarrow \begin{cases} y=1-3x^2 \\ 5x^2=1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x=\frac{1}{\sqrt5} \\ y=\frac{2}{5} \end{cases} e \begin{cases} x=-\frac{1}{\sqrt5} \\ y=\frac{2}{5} \end{cases}.$$

$P_1\equiv(0;0)$; $P_2\equiv(1;0)$; $P_3\equiv(-1;0)$; $P_4\equiv(0;1)$;
$P_5\equiv\left(\frac{1}{\sqrt5};\frac{2}{5}\right)$; $P_6\equiv\left(-\frac{1}{\sqrt5};\frac{2}{5}\right)$.

$$H=\begin{Vmatrix}-6xy & 1-3x^2-2y\\ 1-3x^2-2y & -2x\end{Vmatrix};\ H(0;0)=\begin{Vmatrix}0&1\\1&0\end{Vmatrix}:|H_2|=-1<0:\text{ Saddle.}$$

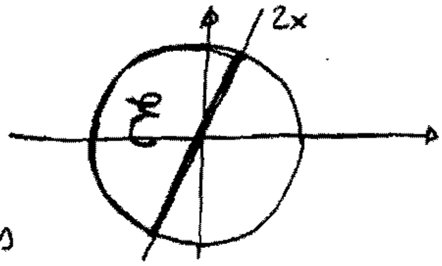
$\boxed{\text{MfEA3}}$

$$H(1;0)=\begin{Vmatrix}0&-2\\-2&-2\end{Vmatrix}:|H_2|=-4<0:\text{ Saddle. } H(-1;0)=\begin{Vmatrix}0&-2\\-2&2\end{Vmatrix}:|H_2|=-4<0:\text{ Saddle.}$$

$$H(0;1)=\begin{Vmatrix}0&-1\\-1&0\end{Vmatrix}:|H_2|=-1<0:\text{ Saddle.}$$

$$H\left(\frac{1}{\sqrt5};\frac25\right)=\begin{Vmatrix}-\frac{12}{5\sqrt5}&-\frac25\\-\frac25&-\frac{2}{\sqrt5}\end{Vmatrix}\Rightarrow\begin{cases}-\frac{12}{5\sqrt5}<0;\ -\frac{2}{\sqrt5}<0\\ \frac{24}{25}-\frac{4}{25}>0\end{cases}:\underline{\text{MAX}}.\ H\left(-\frac{1}{\sqrt5};\frac25\right)=\begin{Vmatrix}\frac{12}{5\sqrt5}&-\frac25\\-\frac25&\frac{2}{\sqrt5}\end{Vmatrix}\Rightarrow\begin{cases}\frac{12}{5\sqrt5}>0;\ \frac{2}{\sqrt5}>0\\ \frac{24}{25}-\frac{4}{25}>0\end{cases}:\underline{\text{MIN}}$$

II M3) $\begin{cases}\text{Max/min } f(x;y)=x^2-y^2\\ \text{s.t.}\begin{cases}x^2+y^2\le1\\ y\ge2x\end{cases}\Rightarrow\begin{cases}x^2+y^2-1\le0\\2x-y\le0\end{cases}\end{cases}$



$\mathcal{E}$ is a closed and bounded set. $f(x;y)$ is a continuous function; constraints are qualified.

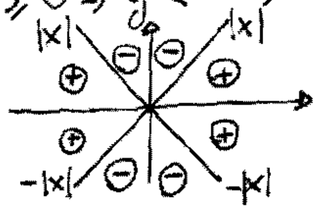
$$\Lambda=x^2-y^2-\lambda_1(x^2+y^2-1)-\lambda_2(2x-y)$$

$\underline{\lambda_1=\lambda_2=0}$

$$\begin{cases}\Lambda'_x=2x=0\\ \Lambda'_y=-2y=0\\ x^2+y^2\le1\\ y\ge2x\end{cases}\Rightarrow\begin{cases}x=0\\y=0\\0\le1\\0\ge0\end{cases}.\ H=\begin{Vmatrix}2&0\\0&-2\end{Vmatrix}:(0;0)\text{ is a saddle point.}$$

$f: x^2-y^2\ge0\Rightarrow y^2\le x^2\Rightarrow -|x|\le y\le|x|$

As $f(0;0)=0$ we study the sign of f:



$\underline{\lambda_1\ne0;\ \lambda_2=0}$

$$\begin{cases}\Lambda'_x=2x-2\lambda_1x=2x(1-\lambda_1)=0\\ \Lambda'_y=-2y-2\lambda_1y=-2y(1+\lambda_1)=0\\ x^2+y^2=1\\ y\ge2x\end{cases}\Rightarrow\begin{cases}x=0\\y=0\\0+0=1\\ \text{NO}\end{cases}\cup\begin{cases}x=0\\ \lambda_1=-1\\ y^2=1\\ y\ge2x\end{cases}\Rightarrow\begin{cases}x=0\\ \lambda_1=-1\\ y=1\\ 1\ge0\end{cases}\cup\begin{cases}x=0\\ \lambda_1=-1\\ y=-1\\ -1\ge0\end{cases}\cup$$

(min?) $\notin\mathcal{E}$

$$\cup\begin{cases}\lambda_1=1\\ y=0\\ x^2=1\\ y\ge2x\end{cases}\Rightarrow\begin{cases}\lambda_1=1\\ y=0\\ x=1\\ 0\not\ge2\end{cases}\cup\begin{cases}\lambda_1=1\\ y=0\\ x=-1\\ 0\ge-2\end{cases}\cup\begin{cases}\lambda_1=1\\ \lambda_1=-1\\ \text{Impossible.}\end{cases}$$

$\notin\mathcal{E}$ (Max?)

MfEA4

$\underline{\lambda_1 = 0;\ \lambda_2 \neq 0}$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_2 = 0 \\ \Lambda'_y = -2y + \lambda_2 = 0 \\ y = 2x \\ x^2+y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} x = \lambda_2 \\ y = \frac{1}{2}\lambda_2 \\ \frac{1}{2}\lambda_2 = 2\lambda_2 \Rightarrow \lambda_2 = 0 \\ x^2+y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} \lambda_2 = 0 \\ x = 0 \\ y = 0 \\ \text{Saddle.} \end{cases}$$

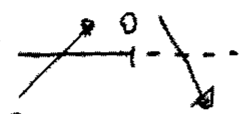
$\underline{\lambda_1 \neq 0;\ \lambda_2 \neq 0}$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x - 2\lambda_2 = 0 \\ \Lambda'_y = -2y - 2\lambda_1 y + \lambda_2 = 0 \\ \begin{cases} y = 2x \\ x^2+y^2 = 1 \end{cases} \Rightarrow \begin{cases} y = 2x \\ 5x^2 = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{5}} \\ y = \frac{2}{\sqrt{5}} \end{cases} \cup \begin{cases} x = -\frac{1}{\sqrt{5}} \\ y = -\frac{2}{\sqrt{5}} \end{cases}. \end{cases}$$

$$\begin{cases} x = \frac{1}{\sqrt{5}} \\ y = \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{5}}\lambda_1 - 2\lambda_2 = 0 \\ -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}}\lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{5}} \\ y = \frac{2}{\sqrt{5}} \\ \lambda_1 = -\frac{3}{5} \\ \lambda_2 = \frac{8}{5\sqrt{5}} \end{cases} ; \quad \begin{cases} x = -\frac{1}{\sqrt{5}} \\ y = -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\lambda_1 + 2\lambda_2 = 0 \\ \frac{4}{\sqrt{5}} + \frac{4}{\sqrt{5}}\lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{\sqrt{5}} \\ y = -\frac{2}{\sqrt{5}} \\ \lambda_1 = -\frac{3}{5} \\ \lambda_2 = -\frac{8}{5\sqrt{5}} \end{cases}$$

nothing. min ?

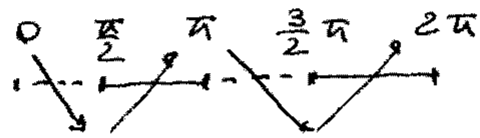
If $y = 2x$: $f(x;2x) = x^2 - 4x^2 = -3x^2$; $f'(x) = -6x > 0$ if $x < 0$:



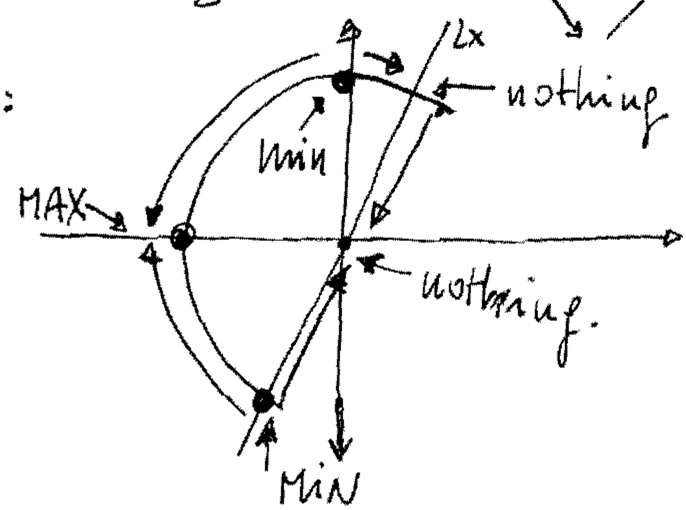
If $x^2+y^2 = 1 \Rightarrow \begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases} \Rightarrow f(x;y) = f(t) = \cos^2 t - \operatorname{sen}^2 t = \cos 2t.$

$f'(t) = -2\operatorname{sen} 2t \geq 0$ if $\operatorname{sen} 2t \leq 0 \Rightarrow \pi \leq 2t \leq 2\pi$ and $3\pi \leq 2t \leq 4\pi \Rightarrow$

$\Rightarrow \frac{\pi}{2} \leq t \leq \pi$ and $\frac{3}{2}\pi \leq t \leq 2\pi$:



And so:



$f(-1;0) = 1$: MAX

$f\left(-\frac{1}{\sqrt{5}}; -\frac{2}{\sqrt{5}}\right) = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$

relative minimum

$f(0;1) = -1$

absolute maximum.

II M4) $f(x;y) = \begin{cases} \dfrac{xy}{x^2+y^2} & : (x;y) \neq (0;0) \\ 0 & : (x;y) = (0;0) \end{cases}$ HfEA5

$$\mathcal{D}_v f(0;0) = \lim_{t \to 0} \frac{f((0;0)+tv) - f(0;0)}{t} .$$ If $v = (t\cos\alpha; t\,\text{sen}\,\alpha) \Rightarrow$

$$\Rightarrow \lim_{t \to 0} \frac{f((0;0)+t(\cos\alpha;\text{sen}\,\alpha)) - 0}{t} = \lim_{t \to 0} \frac{f(\cos\alpha;\text{sen}\,\alpha)}{t} =$$

$$= \lim_{t \to 0} \frac{t^2 \cos\alpha\, \text{sen}\,\alpha}{t^2 \cdot t} = \lim_{t \to 0} \frac{1}{t} \cdot \cos\alpha\, \text{sen}\,\alpha .$$

The limit exists iff $\cos\alpha \cdot \text{sen}\,\alpha = 0 \Rightarrow \alpha = 0;\ \alpha = \frac{\pi}{2};\ \alpha = \pi;\ \alpha = \frac{3}{2}\pi$ and results : $\mathcal{D}_v f(0;0) = 0$; i.e. at the point $(0;0)$ only partial derivatives exist.