Task Mathematics for Economic Applications 19/2/2013 $\boxed{\text{MfEA1}}$

IM1) $z^4 - z^3 + z^2 + 9z - 10 = 0$. If $z = 1$: $1-1+1+9-10=0 \Rightarrow$

$$\begin{array}{c|cccc|c} & 1 & -1 & 1 & 9 & -10 \\ 1 & & 1 & 0 & 1 & 10 \\ \hline & 1 & 0 & 1 & 10 & 0 \end{array} \Rightarrow (z-1)\cdot(z^3+z+10)=0. \text{ If } z=-2: -8-2+10=0 \Rightarrow$$

$$\begin{array}{c|ccc|c} & 1 & 0 & 1 & 10 \\ -2 & & -2 & 4 & -10 \\ \hline & 1 & -2 & 5 & 0 \end{array} \Rightarrow (z-1)(z+2)(z^2-2z+5)=0 \Rightarrow z^2-2z+5=0 \Rightarrow$$

$\Rightarrow z = 1 \pm \sqrt{1-5} = 1 \pm \sqrt{-4} = 1 \pm 2i.$

$z_1+z_2+z_3+z_4 = 1+(-2)+(1+2i)+(1-2i) = 1.$

$1 = 1\cdot(\cos 0 + i \operatorname{sen} 0)$; $\sqrt[4]{1} = 1\cdot\left(\cos k\cdot\frac{2\pi}{4} + i \operatorname{sen} k\cdot\frac{2\pi}{4}\right)$; $0 \le k \le 3$.

$\underline{k=0}$: $\cos 0 + i \operatorname{sen} 0 = 1$; $\underline{k=1}$: $\cos\frac{\pi}{2} + i \operatorname{sen}\frac{\pi}{2} = i$;

$\underline{k=2}$: $\cos\pi + i\operatorname{sen}\pi = -1$; $\underline{k=3}$: $\cos\frac{3}{2}\pi + i \operatorname{sen}\frac{3}{2}\pi = -i$.

IM2) $\begin{cases} x+y-2z=1 \\ 2x-3y+z=-1 \\ x+6y-7z=m \\ 5y-5z=k \end{cases} \Rightarrow \left\|\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 2 & -3 & 1 & -1 \\ 1 & 6 & -7 & m \\ 0 & 5 & -5 & k \end{array}\right\| \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left\|\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -5 & 5 & -3 \\ 0 & 5 & -5 & m-1 \\ 0 & 5 & -5 & k \end{array}\right\| \rightarrow$

$\xrightarrow[R_4 \leftarrow R_4 + R_2]{R_3 \leftarrow R_3 + R_2} \left\|\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -5 & 5 & -3 \\ 0 & 0 & 0 & m-4 \\ 0 & 0 & 0 & k-3 \end{array}\right\|$

•) If $m=4$ and $k=3$: $\text{RANK}(A) = 2 = \text{RANK}(A|Y)$ and the system has ∞^1 solutions.

••) If $m \neq 4$ OR $k \neq 3$: $\text{RANK}(A) = 2 < 3 = \text{RANK}(A|Y)$ and the system has no solutions.

If $m=4$ and $k=3$: $\begin{cases} x+y-2z=1 \\ -5y+5z=-3 \end{cases} \Rightarrow \begin{cases} x = 1-y+2z = z+\frac{2}{5}. \\ y = z+\frac{3}{5} \end{cases}$

IM3) $X = \left\|\begin{matrix} 2 & -1 \\ 1 & -2 \end{matrix}\right\| \cdot \left\|\begin{matrix} -1 \\ 2 \end{matrix}\right\| = \left\|\begin{matrix} -2 & -2 \\ -1 & -4 \end{matrix}\right\| = \left\|\begin{matrix} -4 \\ -5 \end{matrix}\right\| \Rightarrow \left\|\begin{matrix} -4 \\ -5 \end{matrix}\right\| = \left\|\begin{matrix} 3 & 1 \\ 2 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\| \Rightarrow$

$\Rightarrow \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\| = \left\|\begin{matrix} 3 & 1 \\ 2 & 1 \end{matrix}\right\|^{-1} \cdot \left\|\begin{matrix} -4 \\ -5 \end{matrix}\right\| \Rightarrow \left[\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1; \left\|\begin{matrix} 3 & 1 \\ 2 & 1 \end{matrix}\right\| \rightarrow \left\|\begin{matrix} 1 & -2 \\ -1 & 3 \end{matrix}\right\| \rightarrow \left\|\begin{matrix} 1 & -1 \\ -2 & 3 \end{matrix}\right\|\right]$

$\Rightarrow \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\| = \left\|\begin{matrix} 1 & -1 \\ -2 & 3 \end{matrix}\right\| \cdot \left\|\begin{matrix} -4 \\ -5 \end{matrix}\right\| = \left\|\begin{matrix} -4+5 \\ 8-15 \end{matrix}\right\| = \left\|\begin{matrix} 1 \\ -7 \end{matrix}\right\|.$

IM4) A is a symmetric matrix $\Rightarrow$ A is diagonalizable.

$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda-1 & 1 & 1 \\ 0 & -\lambda & 1 \\ 1+\lambda & 1 & -\lambda \end{vmatrix} = -(1+\lambda)\cdot(\lambda^2-1)+(1+\lambda)(1+\lambda) = -(1+\lambda)(\lambda^2-1-1-\lambda) =$

$= -(1+\lambda)(\lambda^2-\lambda-2) = -(1+\lambda)(\lambda+1)(\lambda-2) = 0 \Rightarrow$

$\Rightarrow \lambda_1 = \lambda_2 = -1$; $\lambda_3 = 2$.

If $\lambda = -1$: $\|A-(-1)\mathbb{I}\| = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix} \Rightarrow x+y+z=0 \Rightarrow z = -x-y$.

MfEA2

Eigenvectors: $(x; y; -x-y) \to (x=1; y=0) \to (1; 0; -1)$.

We need another eigenvector, orthogonal to $(1;0;-1)$, so:

$(x; y; -x-y) \cdot (1; 0; -1) = x + x + y = 0 \Rightarrow y = -2x$.

Eigenvectors: $(x; -2x; x) \to (x=1) \to (1; -2; 1)$.

If $\lambda = 2$: $\|A - 2\mathbb{I}\| = \begin{Vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{Vmatrix} \to \begin{Vmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{Vmatrix} \to \begin{Vmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{Vmatrix} \to \begin{Vmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{Vmatrix} \Rightarrow \begin{cases} x-2y+z=0 \\ y-z=0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x-z=0 \\ z=y \end{cases} \Rightarrow \begin{cases} z=x \\ y=x \end{cases}$ so eigenvectors: $(x; x; x) \to (x=1) \to (1;1;1)$.

$$\left.\begin{aligned} &\|1;0;-1\| = \sqrt{2} \ : \text{unit vector}: \left(\tfrac{1}{\sqrt{2}}; 0; -\tfrac{1}{\sqrt{2}}\right) \\ &\|1;-2;1\| = \sqrt{6} \ : \text{unit vector}: \left(\tfrac{1}{\sqrt{6}}; -\tfrac{2}{\sqrt{6}}; \tfrac{1}{\sqrt{6}}\right) \\ &\|1;1;1\| = \sqrt{3} \ : \text{unit vector}: \left(\tfrac{1}{\sqrt{3}}; \tfrac{1}{\sqrt{3}}; \tfrac{1}{\sqrt{3}}\right) \end{aligned}\right\} \Rightarrow U = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{Vmatrix}.$$

II M 1) $f(x;y) = x^3 - 3xy$; $v = \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$.

f is a polynomial, so it is twice differentiable, and so:

$\mathcal{D}_v f = \nabla f \cdot v$ and $\mathcal{D}^2_{v,v} f = v \cdot H \cdot v^T$.

$\nabla f = (3x^2 - 3y; -3x) \Rightarrow \mathcal{D}_v f = (3x^2-3y; -3x) \cdot \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = 0 \Rightarrow$

$\Rightarrow \frac{1}{\sqrt{2}}(3x^2 - 3y - 3x) = \frac{3}{\sqrt{2}}(x^2 - x - y) = 0 \Rightarrow y = x^2 - x$.

$\mathcal{D}^2_{v,v} f = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} 6x & -3 \\ -3 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Vmatrix} = \frac{1}{2} \cdot 6x + \frac{1}{2}(-6) + \frac{1}{2} \cdot 0 = 3x - 3 = 0$ if $x = 1$.

If $x = 1 \Rightarrow y = x^2 - x = 1 - 1 = 0$. So the point is $(x_0; y_0) = (1; 0)$.

II M 2) $f(x;y;z) = x^3y - y^3z + xyz^3 = 0$; $f(1;0;1) = 0 - 0 + 0 = 0$.

$\nabla f = (3x^2y + yz^3; x^3 - 3y^2z + xz^3; -y^3 + 3xyz^2) \Rightarrow \nabla f(1;0;1) = (0;2;0)$.

Being only $\frac{\partial f}{\partial y} \neq 0$ we can define only an implicit function: $(x;z) \to y(x;z)$.

It is: $\frac{\partial y}{\partial x} = -\frac{0}{2} = 0$; $\frac{\partial y}{\partial z} = -\frac{0}{2} = 0$, and so the equation of the tangent plan at point $(x;z) = (1;1)$ is:

$y - y_0 = y'_x \cdot (x-1) + y'_z \cdot (z-1) \Rightarrow y - 0 = 0(x-1) + 0 \cdot (z-1) \Rightarrow y = 0$.

II M3) $\begin{cases} \text{Max/min } f(x;y;z) = x\cdot y\cdot z \\ \text{s.t.}: x+y+z = 5. \end{cases}$

$\boxed{\text{MfEA 3}}$

Using $x+y+z=5$ we can solve: $z = 5-x-y$ and so,

substituting: $f(x;y) = x\cdot y(5-x-y) = 5xy - x^2y - xy^2$.

We check for free maxima and minima of a function $f: \mathbb{R}^2 \to \mathbb{R}$.

$$\begin{cases} f'_x = 5y - 2xy - y^2 = 0 \\ f'_y = 5x - x^2 - 2xy = 0 \end{cases} \Rightarrow \begin{cases} y(5-2x-y) = 0 \\ x(5-x-2y) = 0 \end{cases} \Rightarrow \begin{cases} y=0 \\ x=0 \end{cases} \cup \begin{cases} y=0 \\ 5-x=0 \end{cases} \Rightarrow \begin{cases} y=0 \\ x=5 \end{cases} \cup$$

$$\cup \begin{cases} 5-y=0 \\ x=0 \end{cases} \Rightarrow \begin{cases} y=5 \\ x=0 \end{cases} \cup \begin{cases} y=5-2x \\ 5-x-10+4x = 3x-5=0 \end{cases} \Rightarrow \begin{cases} y = 5-\frac{10}{3} = \frac{5}{3} \\ x = \frac{5}{3} \end{cases}.$$

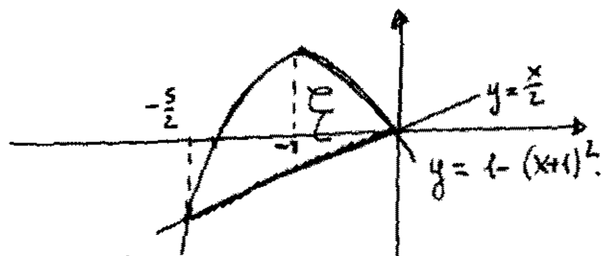
Stationary points: $(0;0)$; $(5;0)$; $(0;5)$; $\left(\frac{5}{3};\frac{5}{3}\right)$. $\quad H = \begin{Vmatrix} -2y & 5-2x-2y \\ 5-2x-2y & -2x \end{Vmatrix}$.

$H(0;0) = \begin{Vmatrix} 0 & 5 \\ 5 & 0 \end{Vmatrix}$: $|H_2| = -25 < 0$: Saddle; $H(5;0) = \begin{Vmatrix} 0 & -5 \\ -5 & -10 \end{Vmatrix}$: $|H_2| = -25 < 0$: Saddle;

$H(0;5) = \begin{Vmatrix} -10 & -5 \\ -5 & 0 \end{Vmatrix}$: $|H_2| = -25 < 0$: Saddle; $H\left(\frac{5}{3};\frac{5}{3}\right) = \begin{Vmatrix} -\frac{10}{3} & -\frac{5}{3} \\ -\frac{5}{3} & -\frac{10}{3} \end{Vmatrix}$: $\begin{matrix} |H_1| < 0 \\ |H_2| > 0 \end{matrix}$: Maximum Point.

If $x = y = \frac{5}{3} \Rightarrow z = 5-x-y = \frac{5}{3}$ so $\left(\frac{5}{3};\frac{5}{3};\frac{5}{3}\right)$ is a maximum Point.

II M4) $\begin{cases} \text{Max/min } f(x;y) = x - xy + y \\ \text{s.t. } \begin{cases} (x+1)^2 + y \leq 1 \\ x - 2y \leq 0 \end{cases} \end{cases}$.



$\mathcal{E}$ is bounded and closed; $f(x;y)$ is a continuous functions; constraints are qualified.

$$\begin{cases} y = \frac{x}{2} \\ y = 1-(x+1)^2 \end{cases} \Rightarrow \begin{cases} y = \frac{x}{2} \\ \frac{x}{2} = 1 - x^2 - 1 - 2x \end{cases} \Rightarrow \begin{cases} y = \frac{x}{2} \\ x^2 + \frac{5}{2}x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x = -\frac{5}{2} \\ y = -\frac{5}{4} \end{cases}.$$

$$\Lambda = x - xy + y - \lambda_1\left((x+1)^2 + y - 1\right) - \lambda_2(x - 2y).$$

<u>Case $\lambda_1 = 0$; $\lambda_2 = 0$</u>

$$\begin{cases} \Lambda'_x = 1 - y = 0 \\ \Lambda'_y = -x + 1 = 0 \\ y \leq 1 - (x+1)^2 \\ 2y \geq x \end{cases} \Rightarrow \begin{cases} y = 1 \\ x = 1 \\ 1 \leq 1 - 4: \text{NO} \\ \quad \notin \mathcal{E}. \end{cases}$$

MfEA4

Case $\lambda_1 \neq 0;\ \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = 1 - y - 2\lambda_1(x+1) = 0 \\ \Lambda'_y = -x + 1 - \lambda_1 = 0 \\ y = 1 - (x+1)^2 \\ 2y \geq x \end{cases} \Rightarrow \begin{cases} x = 1 - \lambda_1 \\ y = 1 - 2\lambda_1(1 - \lambda_1 + 1) = 1 - 2\lambda_1(2 - \lambda_1) \\ 1 - 2\lambda_1(2-\lambda_1) = 1 - (2-\lambda_1)^2 \\ 2y \geq x \end{cases} \Rightarrow$$

$$\Rightarrow 1 - 4\lambda_1 + 2\lambda_1^2 - 1 + 4 + \lambda_1^2 - 4\lambda_1 = 3\lambda_1^2 - 8\lambda_1 + 4 = 0 \Rightarrow \lambda_1 = \frac{4 \pm \sqrt{16-12}}{3} = \frac{4 \pm 2}{3} = \left\langle \begin{matrix} 2 \\ \frac{2}{3} \end{matrix} \right. \Rightarrow$$

$$\Rightarrow \begin{cases} \lambda_1 = 2 > 0 \\ x = -1 \\ y = 1 \\ 2 \cdot 1 \geq -1 : \text{Yes.} \end{cases} \boxed{\text{Max ?}} \cup \begin{cases} \lambda_1 = \frac{2}{3} \\ x = \frac{1}{3} \\ y = -\frac{7}{9} \\ 2 \cdot \left(-\frac{7}{9}\right) \geq \frac{1}{3} : \text{NO} \end{cases} \notin \mathcal{E}.$$

Case $\lambda_1 \neq 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 1 - y - 2\lambda_1(x+1) - \lambda_2 = 0 \\ \Lambda'_y = -x + 1 - \lambda_1 + 2\lambda_2 = 0 \\ \begin{cases} y = 1 - (x+1)^2 \\ x = 2y \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x = -\frac{5}{2} \\ y = -\frac{5}{4} \end{cases} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 2\lambda_1 + \lambda_2 = 1 \\ \lambda_1 - 2\lambda_2 = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_2 = -\frac{1}{5} < 0 : \text{Nothing.} \\ \lambda_1 = \frac{3}{5} > 0 \end{cases}$$

$$\begin{cases} x = -\frac{5}{2} \\ y = -\frac{5}{4} \\ 1 + \frac{5}{4} + 3\lambda_1 - \lambda_2 = 0 \\ \frac{5}{2} + 1 - \lambda_1 + 2\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{5}{2} \\ y = -\frac{5}{4} \\ 3\lambda_1 - \lambda_2 = -\frac{9}{4} \\ \lambda_1 - 2\lambda_2 = \frac{7}{2} \end{cases} \Rightarrow \begin{cases} x = -\frac{5}{2} \\ y = -\frac{5}{4} \\ \lambda_1 = -\frac{8}{5} < 0 \\ \lambda_2 = -\frac{51}{20} < 0 \end{cases} \boxed{\text{Min ?}}$$

Since hypoteses of Weierstrass Theorem are satisfied, and since we have only one solution for Maximum and only one solution for minimum, point $(-1;1)$ is the maximum point, with $f(-1;1) = 1$ and point $\left(-\frac{5}{2}; -\frac{5}{4}\right)$ is the minimum point, with $f\left(-\frac{5}{2}; -\frac{5}{4}\right) = -\frac{55}{8}$.