Task Mathematics for Economic Applications 23/3/2013 — MfEA1

I M1) $(1+i) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\cdot\left(\cos\frac{\pi}{4} + i\,\mathrm{sen}\,\frac{\pi}{4}\right)$.

$(1+i)^{20} = (\sqrt{2})^{20}\cdot\left(\cos\left(20\cdot\frac{\pi}{4}\right) + i\,\mathrm{sen}\left(20\cdot\frac{\pi}{4}\right)\right) = 2^{10}\cdot(\cos 5\pi + i\,\mathrm{sen}\,5\pi) = 2^{10}\cdot(-1) = -2^{10}$.

$(1-i) = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\cdot\left(\cos\frac{7}{4}\pi + i\,\mathrm{sen}\,\frac{7}{4}\pi\right)$.

$(1-i)^{12} = (\sqrt{2})^{12}\cdot\left(\cos\left(12\cdot\frac{7}{4}\pi\right) + i\,\mathrm{sen}\left(12\cdot\frac{7}{4}\pi\right)\right) = 2^{6}\cdot(\cos 21\pi + i\,\mathrm{sen}\,21\pi) = 2^{6}\cdot(-1) = -2^{6}$.

$\sqrt{(1+i)^{20} - (1-i)^{12}} = \sqrt{-2^{10}+2^{6}} = \sqrt{2^{6}\cdot(1-2^{4})} = 2^{3}\cdot\sqrt{1-16} = 8\cdot\sqrt{-15} = 8\cdot\sqrt{15}\cdot\sqrt{-1} = \pm 8\cdot\sqrt{15}\cdot i$.

I M2) B similar to A with P $\Rightarrow A\cdot P = P\cdot B \Rightarrow B = P^{-1}\cdot A\cdot P$.

$P = \begin{Vmatrix} 3 & 1 \\ 5 & 2 \end{Vmatrix}$; $|P| = 6-5 = 1$; $\begin{Vmatrix} 3 & 1 \\ 5 & 2 \end{Vmatrix} \xrightarrow{A_t} \begin{Vmatrix} 2 & -5 \\ -1 & 3 \end{Vmatrix} \xrightarrow{T} \begin{Vmatrix} 2 & -1 \\ -5 & 3 \end{Vmatrix} = P^{-1}$.

$B = \begin{Vmatrix} 2 & -1 \\ -5 & 3 \end{Vmatrix}\cdot\begin{Vmatrix} 4 & -1 \\ 2 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} 3 & 1 \\ 5 & 2 \end{Vmatrix} = \begin{Vmatrix} 2 & -1 \\ -5 & 3 \end{Vmatrix}\cdot\begin{Vmatrix} 7 & 2 \\ 11 & 4 \end{Vmatrix} = \begin{Vmatrix} 3 & 0 \\ -2 & 2 \end{Vmatrix} = B$.

$\begin{vmatrix} 3-\lambda & 0 \\ -2 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda_1 = 2 ; \lambda_2 = 3$.

$\|B - 2\mathbb{I}\| = \begin{Vmatrix} 1 & 0 \\ -2 & 0 \end{Vmatrix} \Rightarrow x + 0y = 0 \Rightarrow x = 0$: Eigenvector for $\lambda_1 = 2$: $(0;1)$.

$\|B - 3\mathbb{I}\| = \begin{Vmatrix} 0 & 0 \\ -2 & -1 \end{Vmatrix} \Rightarrow -2x - y = 0 \Rightarrow y = -2x$: Eigenvector for $\lambda_2 = 3$: $(1;-2)$.

Basis for $\mathbb{R}^2$: $W_B = \{(0;1);(1;-2)\}$.

I M3) $\begin{Vmatrix} 1 & 0 & 0 & k \\ 0 & 1 & 0 & k \\ m & 0 & 1 & 0 \\ m & 0 & 0 & 1 \end{Vmatrix} \to \begin{Vmatrix} 1 & 0 & 0 & k \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 1-mk \\ 0 & 0 & 0 & 1-mk \end{Vmatrix} \Rightarrow \mathrm{RANK}(A) = \begin{cases} 3 \text{ if } 1-mk = 0 \Rightarrow k = \frac{1}{m} \\ 4 \text{ if } 1-mk \neq 0. \end{cases}$

For $k = \frac{1}{m}$: $\begin{Vmatrix} 1 & 0 & 0 & \frac{1}{m} \\ 0 & 1 & 0 & \frac{1}{m} \\ m & 0 & 1 & 0 \\ m & 0 & 0 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1+\frac{1}{m} \\ 1+\frac{1}{m} \\ m+1 \\ m+1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}$ iff $m = -1 \Rightarrow k = -1$.

I M4) $\begin{Vmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{Vmatrix} \to \begin{vmatrix} 2-\lambda & 2 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)\left((2-\lambda)^2 - 1\right) = (1-\lambda)(\lambda^2 - 4\lambda + 3) = 0 \Rightarrow$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-3) = 0 \Rightarrow \lambda_1 = \lambda_2 = 1 ; \lambda_3 = 3$.

For $\lambda = 3$: $\begin{Vmatrix} -1 & 2 & 1 \\ 0 & -2 & 0 \\ 1 & 2 & -1 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \underline{0} \Rightarrow \begin{cases} -x+2y+z = 0 \\ -2y = 0 \\ x+2y-z = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ z = x \\ x+0-x = 0 \end{cases} \Rightarrow (x;0;x) \Rightarrow (1;0;1)$.

For $\lambda = 1$: $\begin{Vmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \underline{0} \Rightarrow x + 2y + z = 0 \Rightarrow z = -x - 2y \Rightarrow (x;y;-x-2y)$.

$(1;0;1)\cdot(x;y;-x-2y) = x - x - 2y = 0$ iff $y = 0 \Rightarrow (x;0;-x)$.

From $(x;0;-x)$ we can get only one vector orthogonal to $(1;0;1)$, while instead we need to vectors. So it is impossible to diagonalize A with an orthogonal matrix.

II M1) $f(x;y) = x^4 - x^2y - e^y = 0$.

$\nabla f(x;y) = (4x^3 - 2xy ; -x^2 - e^y)$. Since $(-x^2 - e^y) \neq 0\ \forall (x;y) \in \mathbb{R}^2$ it is always possible to define an implicit function $x \to y = y(x)$. Given $P_0 = (1;0)$:

$$\nabla f(1;0) = (4-0;\,-1-1) = (4;\,-2) \Rightarrow y'(1) = -\frac{4}{-2} = 2.$$

$\boxed{\text{MfEA2}}$

$$H(f) = \begin{Vmatrix} 12x^2-2y & -2x \\ -2x & -e^y \end{Vmatrix};\; H(f(1;0)) = \begin{Vmatrix} 12 & -2 \\ -2 & -1 \end{Vmatrix}.$$

$$y''(1) = -\frac{12+2(-2)\cdot y' + (-1)(y')^2}{-2} = -\frac{12-8-4}{-2} = 0.$$

II M2) $f(x;y) = xe^y + ye^x$: continuous and twice differentiable function.

$\nabla f(x;y) = (e^y + ye^x;\, xe^y + e^x)$. $v = (\cos\alpha;\, \operatorname{sen}\alpha)$.

$D_v f(0;0) = (1+0;\, 0+1)\cdot(\cos\alpha;\, \operatorname{sen}\alpha) = \cos\alpha + \operatorname{sen}\alpha = 0 \Rightarrow \cos\alpha = -\operatorname{sen}\alpha : \alpha = \frac{3}{4}\pi \,;\, \frac{7}{4}\pi.$

$$H(f) = \begin{Vmatrix} ye^x & e^y+e^x \\ e^y+e^x & xe^y \end{Vmatrix};\; D^2_{v,v} f(0;0) = \begin{Vmatrix} \cos\alpha & \operatorname{sen}\alpha \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 2 \\ 2 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\alpha \\ \operatorname{sen}\alpha \end{Vmatrix} = 4 \operatorname{sen}\alpha \cos\alpha = 0.$$

$4\operatorname{sen}\alpha\cos\alpha = 2\cdot\operatorname{sen}2\alpha = 0 \Rightarrow \operatorname{sen}2\alpha = 0 : 2\alpha = 0 \Rightarrow \alpha = 0$ and $2\alpha = \pi \Rightarrow \alpha = \frac{\pi}{2}$.

There are no directions v such that $D_v f(0;0) = 0$ and $D^2_{v,v} f(0;0) = 0$.

II M3) To verify if the function is differentiable at $(0;0)$ we must get:

$$\lim_{(x;y)\to(0;0)} \frac{f(x;y) - f(0;0) - \nabla f(0;0)\cdot(x-0;\, y-0)}{\sqrt{x^2+y^2}} = 0.$$

The function is continuous $\forall (x;y) \in \mathbb{R}^2$.

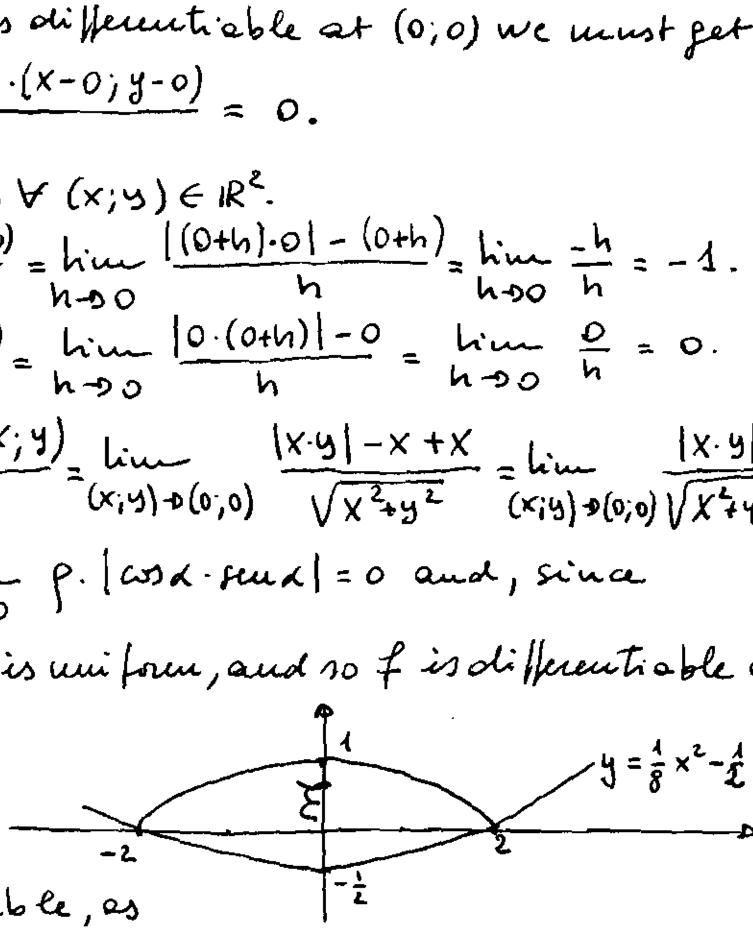
$$\frac{\partial f}{\partial x}(0;0) = \lim_{h\to 0} \frac{f(0+h;0) - f(0;0)}{h} = \lim_{h\to 0} \frac{|(0+h)\cdot 0| - (0+h)}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

$$\frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0} \frac{f(0;0+h) - f(0;0)}{h} = \lim_{h\to 0} \frac{|0\cdot(0+h)| - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

$$\lim_{(x;y)\to(0;0)} \frac{|x\cdot y| - x - 0 - (-1;0)(x;y)}{\sqrt{x^2+y^2}} = \lim_{(x;y)\to(0;0)} \frac{|x\cdot y| - x + x}{\sqrt{x^2+y^2}} = \lim_{(x;y)\to(0;0)} \frac{|x\cdot y|}{\sqrt{x^2+y^2}} \Rightarrow$$

$\Rightarrow \lim\limits_{\rho\to 0} \dfrac{\rho^2 |\cos\alpha\cdot\operatorname{sen}\alpha|}{\rho} = \lim\limits_{\rho\to 0} \rho\cdot|\cos\alpha\cdot\operatorname{sen}\alpha| = 0$ and, since $|\cos\alpha\cdot\operatorname{sen}\alpha| < 1$, the convergence is uniform, and so f is differentiable at $(0;0)$.

II M4) $\begin{cases} \text{Max/min } f(x;y) = x - y \\ \text{s.t.: } \begin{cases} x^2+4y^2 \le 4 \\ x^2 - 8y \le 4 \end{cases} \end{cases}$



f is continuous and differentiable, as constraints. $\mathcal{E}$ is a closed and bounded set, constraints are qualified.

$\Lambda = x - y - \lambda_1(x^2+4y^2-4) - \lambda_2(x^2-8y-4)$

If $\lambda_1 = 0;\, \lambda_2 = 0$

$\Lambda'_x = 1 \neq 0$ there are no solutions.

MfEA3

If $\lambda_1 \neq 0; \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = 1 - 2\lambda_1 x = 0 \\ \Lambda'_y = -1 - 8\lambda_1 y = 0 \\ x^2 + 4y^2 = 4 \\ x^2 - 8y \leq 4 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda_1} \\ y = -\frac{1}{8\lambda_1} \\ \frac{1}{4\lambda_1^2} + 4 \cdot \frac{1}{64\lambda_1^2} = 4 \Rightarrow \lambda_1^2 = \frac{5}{64} \Rightarrow \\ x^2 - 8y \leq 4 \end{cases} \begin{cases} \lambda_1 = \frac{\sqrt{5}}{8} \\ x = \frac{4}{\sqrt{5}} \\ y = -\frac{1}{\sqrt{5}} \\ \frac{16}{5} + \frac{8}{\sqrt{5}} \leq 4 \end{cases} ; \begin{cases} \lambda_1 = -\frac{\sqrt{5}}{8} \\ x = -\frac{4}{\sqrt{5}} \\ y = \frac{1}{\sqrt{5}} \\ \frac{16}{5} - \frac{8}{\sqrt{5}} \leq 4 \end{cases} \boxed{\text{Min ?}}$$

$\notin \mathcal{E}$ not true; True

If $\lambda_1 = 0 ; \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 1 - 2\lambda_2 x = 0 \\ \Lambda'_y = -1 + 8\lambda_2 = 0 \\ x^2 - 8y = 4 \\ x^2 + 4y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} \lambda_2 = \frac{1}{8} \\ x = \frac{1}{2\lambda_2} = 4 \\ y = \frac{1}{8}(x^2 - 4) = \frac{1}{8}(16 - 4) = \frac{3}{2} \\ 16 + 4 \cdot \frac{9}{4} \leq 4 : \text{not true} \end{cases}$$

$\notin \mathcal{E}$

If $\lambda_1 \neq 0 ; \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 1 - 2\lambda_1 x - 2\lambda_2 x = 0 \\ \Lambda'_y = -1 - 8\lambda_1 y + 8\lambda_2 = 0 \\ \begin{cases} x^2 + 4y^2 = 4 \\ x^2 - 8y = 4 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 0 \end{cases} \cup \begin{cases} x = -2 \\ y = 0 \end{cases} \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 0 \\ 1 - 4\lambda_1 - 4\lambda_2 = 0 \\ -1 - 0 + 8\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 0 \\ \lambda_1 = \frac{1}{4}(1 - 4\lambda_2) = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} > 0 \\ \lambda_2 = \frac{1}{8} > 0 \end{cases} \boxed{\text{MAX ?}}$$

$$\begin{cases} x = -2 \\ y = 0 \\ 1 + 4\lambda_1 + 4\lambda_2 = 0 \\ -1 - 0 + 8\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = 0 \\ \lambda_1 = -\frac{1}{4}(1 + 4\lambda_2) = -\frac{1}{4}\left(1 + \frac{1}{2}\right) = -\frac{3}{8} < 0 \\ \lambda_2 = \frac{1}{8} > 0 \end{cases} \quad \text{Nothing.}$$

Having found only one point for minimum and only one point for maximum, these are the two solutions following Weierstrass's Theorem.