Task Mathematics for Economic Applications 11/6/2013 $\boxed{\text{MfEA1}}$

I M1) $z^3-3z^2+4z+8=0$. If $z=-1$: $-1-3-4+8=0$.

$$\begin{array}{c|ccc|c} & 1 & -3 & 4 & 8 \\ -1 & & -1 & 4 & -8 \\ \hline & 1 & -4 & 8 & 0 \end{array} \Rightarrow \begin{array}{l} z^3-3z^2+4z+8=(z+1)(z^2-4z+8)=0. \\ z=2\pm\sqrt{4-8}=2\pm\sqrt{-4}=2\pm 2i. \end{array}$$

$z_1=-1$; $z_2=2+2i$; $z_3=2-2i$.

$z_1\cdot z_2\cdot z_3=-1\cdot(2+2i)(2-2i)=-1\cdot(4+4)=-8=8(\cos\pi+i\operatorname{sen}\pi)$.

$\sqrt[3]{-8}=\sqrt[3]{8}\cdot\left(\cos\left(\frac{\pi}{3}+k\cdot\frac{2\pi}{3}\right)+i\operatorname{sen}\left(\frac{\pi}{3}+k\cdot\frac{2\pi}{3}\right)\right)$; $0\le k\le 2$.

If $k=0$: $2\cdot\left(\cos\frac{\pi}{3}+i\operatorname{sen}\frac{\pi}{3}\right)=2\cdot\left(\frac{1}{2}+i\frac{\sqrt3}{2}\right)=1+i\sqrt3$;

If $k=1$: $2\cdot(\cos\pi+i\operatorname{sen}\pi)=-2$;

If $k=2$: $2\cdot\left(\cos\frac{5}{3}\pi+i\operatorname{sen}\frac{5}{3}\pi\right)=2\left(\frac{1}{2}-i\frac{\sqrt3}{2}\right)=1-i\sqrt3$.

I M2) $\begin{Vmatrix} 2 & 2 & -1 \\ -1 & 2 & k \\ 3 & 0 & -2 \end{Vmatrix} \to \begin{vmatrix} 2-\lambda & 2 & -1 \\ -1 & 2-\lambda & k \\ 3 & 0 & -2-\lambda \end{vmatrix} = 3\cdot\begin{vmatrix} 2 & -1 \\ 2-\lambda & k \end{vmatrix} - (2+\lambda)\cdot\begin{vmatrix} 2-\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} =$

$=3(2k+2-\lambda)-(2+\lambda)(\lambda^2-4\lambda+4+2)=6k+6-3\lambda-2\lambda^2+8\lambda-12-\lambda^3+4\lambda^2-6\lambda=$

$=-\lambda^3+2\lambda^2-\lambda+6k-6=0$; if $\lambda=1$: $-1+2-1+6k-6=0 \Rightarrow k=1$.

$\lambda^3-2\lambda^2+\lambda=0 \Rightarrow \lambda(\lambda^2-2\lambda+1)=\lambda\cdot(\lambda-1)^2=0 \Rightarrow \lambda_1=0$; $\lambda_2=\lambda_3=1$.

If $\lambda=1$: $\|A-1\cdot\mathbb{I}\|=\begin{Vmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 3 & 0 & -3 \end{Vmatrix} \to \begin{Vmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ 0 & -6 & 0 \end{Vmatrix}$: $\text{RANK}(A-1\cdot\mathbb{I})=2 \Rightarrow m^g_1=3-2=1$. So $m^g_1=1<m^a_1=2$.

The matrix is not diagonalisable.

I M3) $(1;-1;2)\in\text{Ker}(A) \Rightarrow \begin{Vmatrix} 1 & 1 & k \\ 0 & m & -1 \end{Vmatrix}\cdot\begin{Vmatrix} 1 \\ -1 \\ 2 \end{Vmatrix}=\begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} 1-1+2k=0 \\ 0-m-2=0 \end{cases} \Rightarrow \begin{cases} k=0 \\ m=-2 \end{cases}$

$\Rightarrow A=\begin{Vmatrix} 1 & 1 & 0 \\ 0 & -2 & -1 \end{Vmatrix} \Rightarrow \text{RANK}(A)=2 \Rightarrow \text{Dim}(\text{Imm})=2$ and $\text{Dim}(\text{Ker})=1$.

I M4) $\left\Vert\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 2 & 1 & -1 & 1 & 2 \\ 1 & 1 & m & 2 & k \end{array}\right\Vert \xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-2R_1} \left\Vert\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & -1 & 3 & -4 \\ 0 & 1 & m & 3 & k-3 \end{array}\right\Vert \xrightarrow{R_3\leftarrow R_3-R_2} \left\Vert\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & -1 & 3 & -4 \\ 0 & 0 & m+1 & 0 & k+1 \end{array}\right\Vert$

If $m=-1$ and $k=-1$: $\text{RANK}(A)=2=\text{RANK}(A|Y)$: ∞^2 Solutions;

If $m=-1$ and $k\neq-1$: $\text{RANK}(A)=2<\text{RANK}(A|Y)=3$: NO Solutions;

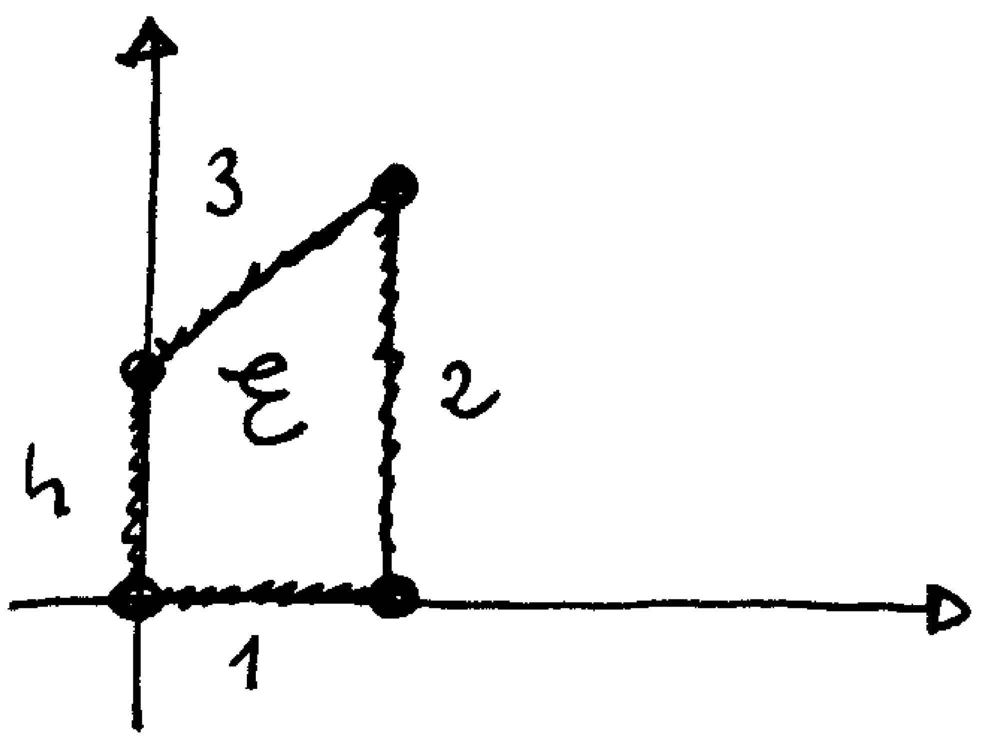
If $m\neq-1$ $\forall k\in\mathbb{R}$: $\text{RANK}(A)=3=\text{RANK}(A|Y)$: ∞^1 Solutions.

II M1) $f(x;y)=3y-2x$: Continuous function

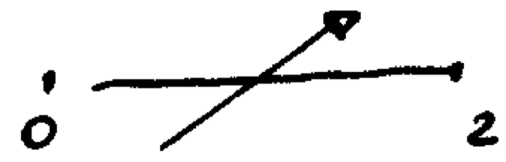
$\mathcal{E}$: bounded and closed set.

f has Maximum and minimum points.

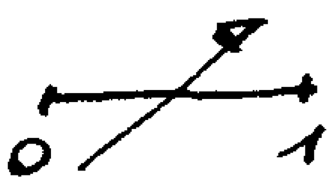
Since $f'_x=-2\neq0$ and $f'_y=3\neq0$ there are no points of free maximum or minimum.

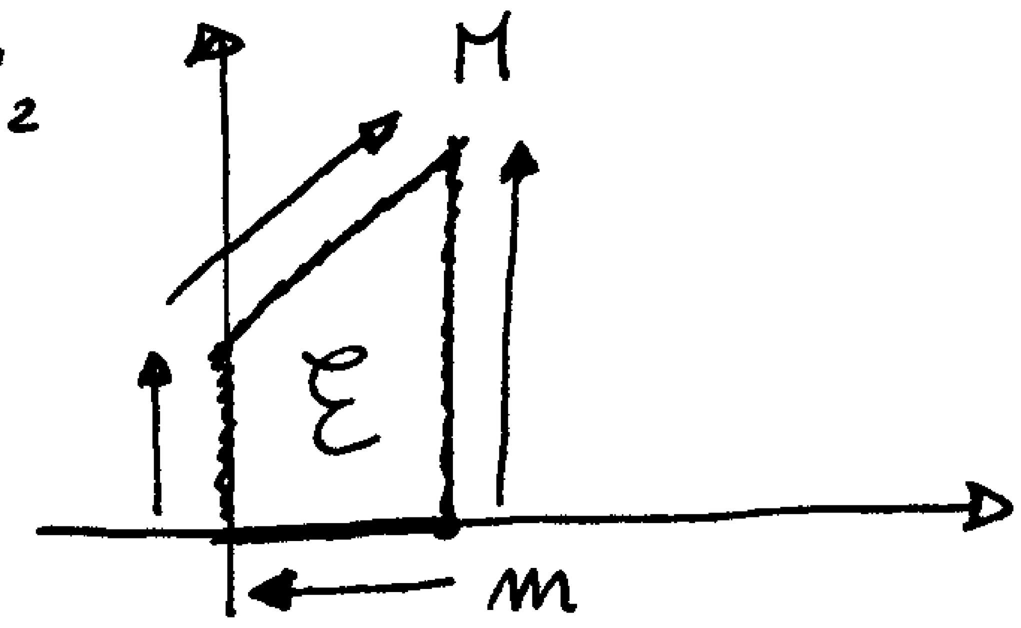


1) If $y=0$: $f(x;0) = -2x$: $f'(x) = -2 < 0$: 

2) If $x=1$: $f(1;y) = 3y-2$: $f'(y) = 3 > 0$: 

3) If $y = x+1$; $f(x;x+1) = x+3$: $f'(x) = 1 > 0$: 

4) If $x=0$: $f(0;y) = 3y$: $f'(y) = 3 > 0$: 



So $(1;0)$ is the minimum point: $f(1;0) = -2$;

$(1;2)$ is the maximum point: $f(1;2) = 4$.

II M 2) $f(x;y) = x^2 - xy + 3y$. f is a differentiable function.

$(1;1) = \sqrt{2}\left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)$; $u = \left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)$; $(1;-1) = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}}\right)$; $v = \left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}}\right)$.

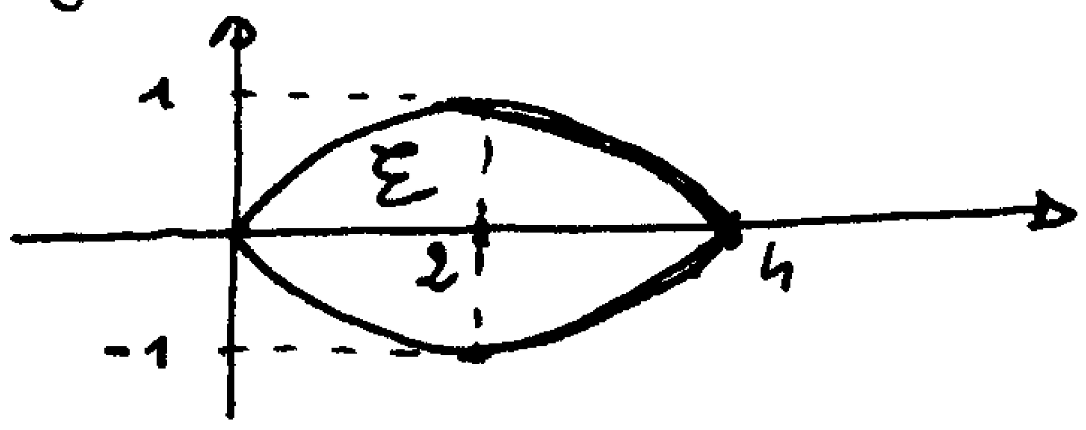
$\nabla f = (2x-y;-x+3)$.

$D_u f(P_0) = (2x-y;-x+3)\cdot\left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(2x-y-x+3) = \frac{1}{\sqrt{2}}(x-y+3) = \sqrt{2}$ $\Rightarrow$

$D_v f(P_0) = (2x-y;-x+3)\cdot\left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(2x-y+x-3) = \frac{1}{\sqrt{2}}(3x-y-3) = -2\sqrt{2}$

$$\Rightarrow \begin{cases} x-y+3 = 2 \\ 3x-y-3 = -4 \end{cases} \Rightarrow \begin{cases} x-y = -1 \\ 3x-y = -1 \end{cases} \Rightarrow \begin{cases} 2x = 0 \\ y = +1 \end{cases} \Rightarrow P_0 = (0;1).$$

II M 3) $\begin{cases} \text{Max/min } f(x;y) = x-2y \\ \text{s.t. : } \frac{(x-2)^2}{4} + y^2 \leq 1 \end{cases}$



$f'_x = 1 \neq 0$; $f'_y = -2 \neq 0$ No free Max or min points.

$\Lambda = x - 2y - \lambda\left(\frac{(x-2)^2}{4} + y^2 - 1\right)$

$$\begin{cases} \Lambda'_x = 1 - \frac{\lambda(x-2)}{2} = 0 \\ \Lambda'_y = -2 - 2\lambda y = 0 \\ \frac{(x-2)^2}{4} + y^2 = 1 \end{cases} \Rightarrow \begin{cases} \lambda(x-2) = 2 \\ -2(1+\lambda y) = 0 \\ \frac{(x-2)^2}{4} + y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{2}{\lambda} + 2 \\ y = -\frac{1}{\lambda} \\ \frac{1}{4}\left(\frac{2}{\lambda}+2-2\right)^2 + \frac{1}{\lambda^2} = 1 \end{cases} \Rightarrow$$

$\Rightarrow \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 1 \Rightarrow \lambda^2 = 2 \Rightarrow \lambda = \pm\sqrt{2}$.

$\begin{cases} \lambda = \sqrt{2} > 0 \\ x = \sqrt{2} + 2 \\ y = -\frac{1}{\sqrt{2}} \end{cases}$ $\boxed{\text{MAX ?}}$ $\begin{cases} \lambda = -\sqrt{2} < 0 \\ x = -\sqrt{2} + 2 \\ y = \frac{1}{\sqrt{2}} \end{cases}$ $\boxed{\text{Min ?}}$

f is a continuous function, Ɛ is a bounded and closed set, so Weierstrass's Theorem holds, and so:

$\left(\sqrt{2}+2;-\frac{1}{\sqrt{2}}\right)$ is the Maximum point; $\left(2-\sqrt{2};\frac{1}{\sqrt{2}}\right)$ is the minimum point.

MfEA3

II M4) $f(x;y) = e^{x-y} - \cos(x-y) = 0$; $f(1;1) = 1-1 = 0$.

$\nabla f(x;y) = \left(e^{x-y} + \operatorname{sen}(x-y) ; -e^{x-y} - \operatorname{sen}(x-y)\right)$.

$\nabla f(1;1) = (1+0 ; -1-0) = (1;-1)$; $f'_y(1;1) \neq 0$.

It is possible to define an implicit function $y = y(x)$.

$y'(1) = -\frac{1}{-1} = 1$.

$$H(f) = \begin{Vmatrix} e^{x-y} + \cos(x-y) & -e^{x-y} - \cos(x-y) \\ -e^{x-y} - \cos(x-y) & e^{x-y} + \cos(x-y) \end{Vmatrix}.$$

$$H(1;1) = \begin{Vmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{Vmatrix} = \begin{Vmatrix} 2 & -2 \\ -2 & 2 \end{Vmatrix}.$$

$$y''(1) = -\frac{2 + 2\cdot(-2)\cdot 1 + 2\cdot(1)^2}{-1} = 2 - 4 + 2 = 0.$$

$$\left(y''(1) = -\frac{f''_{xx} + 2 f''_{xy}\cdot y' + f''_{yy}(y')^2}{f'_y}\right)$$