Task Mathematics for economic applications 5/7/2013 $\boxed{\text{MfEA1}}$

IM1) $\frac{15}{2(1-3i)}-\frac{3}{2(1+i)}=\frac{3}{2}\left(\frac{5}{1-3i}-\frac{1}{1+i}\right)=\frac{3}{2}\left(\frac{5+5i-1+3i}{(1-3i)(1+i)}\right)=\frac{3}{2}\cdot\frac{4+8i}{4-2i}=3\cdot\frac{1+2i}{2-i}=$

$=3\cdot\frac{(1+2i)(2+i)}{(2-i)(2+i)}=3\cdot\frac{0+5i}{4+1}=3i=3\cdot\left(\cos\frac{\pi}{2}+i\,\text{sen}\,\frac{\pi}{2}\right)=z.$

$\sqrt[3]{z}=\sqrt[3]{3}\cdot\left(\cos\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{3}\right)+i\,\text{sen}\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{3}\right)\right);\ 0\le k\le 2.$

For $k=0$: $\sqrt[3]{3}\cdot\left(\cos\frac{\pi}{6}+i\,\text{sen}\,\frac{\pi}{6}\right)=\sqrt[3]{3}\cdot\left(\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right);$

For $k=1$: $\sqrt[3]{3}\cdot\left(\cos\frac{5}{6}\pi+i\,\text{sen}\,\frac{5}{6}\pi\right)=\sqrt[3]{3}\cdot\left(-\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right);$

For $k=2$: $\sqrt[3]{3}\cdot\left(\cos\frac{3}{2}\pi+i\,\text{sen}\,\frac{3}{2}\pi\right)=\sqrt[3]{3}\cdot(-i).$

IM2) $\left\|\begin{matrix}0&k&0\\1&0&0\\0&0&k\end{matrix}\right\|\to\left|\begin{matrix}-\lambda&k&0\\1&-\lambda&0\\0&0&k-\lambda\end{matrix}\right|=(k-\lambda)(\lambda^2-k)=0\Rightarrow\lambda_1=k;\ \lambda_2=\sqrt{k};\ \lambda_3=-\sqrt{k}.$

We have multiple eigenvalues if: $k=0\Rightarrow\lambda_1=\lambda_2=\lambda_3=0$ and if

$k=1\Rightarrow\lambda_1=\lambda_2=1,\ \lambda_3=-1.$

For $k=0$ and $\lambda=0$: $\|A-0\mathbb{I}\|=\left\|\begin{matrix}0&0&0\\1&0&0\\0&0&0\end{matrix}\right\|$; RANK $=1\Rightarrow m_0^g=3-1=2<m_0^a=3$ so the matrix is not diagonalizable.

For $k=1$ and $\lambda=1$: $\|A-1\cdot\mathbb{I}\|=\left\|\begin{matrix}-1&1&0\\1&-1&0\\0&0&0\end{matrix}\right\|$: RANK $=1\Rightarrow m_1^g=3-1=2=m_1^g$ so the matrix is diagonalizable.

The matrix is a diagonalizable one for every $k\in\mathbb{R}$, $k\neq 0$.

IM3) $A=\left\|\begin{matrix}a_1&b_1&c_1\\a_2&b_2&c_2\end{matrix}\right\|$; $f(1;1;1)=(-1;1)$ and $(1;0;1)\in\text{Ker}(A)$.

$\left\|\begin{matrix}a_1&b_1&c_1\\a_2&b_2&c_2\end{matrix}\right\|\cdot\left\|\begin{matrix}1\\1\\1\end{matrix}\right\|=\left\|\begin{matrix}a_1+b_1+c_1\\a_2+b_2+c_2\end{matrix}\right\|=\left\|\begin{matrix}-1\\1\end{matrix}\right\|\Rightarrow\begin{cases}a_1+b_1+c_1=-1\\a_2+b_2+c_2=1\end{cases}$

$\left\|\begin{matrix}a_1&b_1&c_1\\a_2&b_2&c_2\end{matrix}\right\|\cdot\left\|\begin{matrix}1\\0\\1\end{matrix}\right\|=\left\|\begin{matrix}a_1+c_1\\a_2+c_2\end{matrix}\right\|=\left\|\begin{matrix}0\\0\end{matrix}\right\|\Rightarrow\begin{cases}a_1+c_1=0\\a_2+c_2=0\end{cases}\Rightarrow\begin{cases}c_1=-a_1\\c_2=-a_2\end{cases}\Rightarrow\begin{cases}b_1=-1\\b_2=1.\end{cases}$

So $A=\left\|\begin{matrix}a_1&-1&-a_1\\a_2&1&-a_2\end{matrix}\right\|$. RANK$(A)=$ Dim (Imm) $=1$ iff $\frac{a_1}{a_2}=\frac{-1}{1}=\frac{-a_1}{-a_2}\Rightarrow a_2=-a_1$

We have Dim (Imm) $=1$ iff $A=\left\|\begin{matrix}k&-1&-k\\-k&1&k\end{matrix}\right\|$, $\forall k\in\mathbb{R}$.

IM4) $A\cdot X=Y\Rightarrow X=A^{-1}\cdot Y$

$A=\left\|\begin{matrix}1&1&1\\0&1&0\\1&0&0\end{matrix}\right\|;\ X=\left\|\begin{matrix}x_1\\x_2\\x_3\end{matrix}\right\|;\ Y=\left\|\begin{matrix}1\\2\\3\end{matrix}\right\|.$

MfEA2

$$\left\|\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&1&0&0&1&0\\1&0&0&0&0&1\end{array}\right\| \xrightarrow[R_3 \leftarrow R_3 - R_1]{} \left\|\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&1&0&0&1&0\\0&-1&-1&-1&0&1\end{array}\right\| \xrightarrow[R_3 \leftarrow R_3 + R_2]{} \left\|\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&1&0&0&1&0\\0&0&-1&-1&1&1\end{array}\right\| \xrightarrow[R_3 \leftarrow -R_3]{}$$

$$\to \left\|\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&1&0&0&1&0\\0&0&1&1&-1&-1\end{array}\right\| \xrightarrow[R_1 \leftarrow R_1 - R_2 - R_3]{} \left\|\begin{array}{ccc:ccc}1&0&0&0&0&1\\0&1&0&0&1&0\\0&0&1&1&-1&-1\end{array}\right\| \Rightarrow A^{-1} = \begin{Vmatrix}0&0&1\\0&1&0\\1&-1&-1\end{Vmatrix}.$$

$A^{-1} \cdot Y = X \Rightarrow \begin{Vmatrix}0&0&1\\0&1&0\\1&-1&-1\end{Vmatrix} \cdot \begin{Vmatrix}1\\2\\3\end{Vmatrix} = \begin{Vmatrix}3\\2\\-4\end{Vmatrix}$: coordinates of X in the new basis.

II M1) $f(x;y) = x^3 + y^3 - kxy$. $\quad \nabla f(x;y) = (0;0) \Rightarrow$

$$\begin{cases} f'_x = 3x^2 - ky = 0 \\ f'_y = 3y^2 - kx = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{3}{k} \cdot x^2 \\ 3 \cdot \frac{9}{k^2} \cdot x^4 - kx = x \cdot \left(\frac{27}{k^2} x^3 - k\right) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or}$$

$$\begin{cases} x^3 = \frac{k^3}{27} \Rightarrow x = \frac{k}{3} \\ y = \frac{3}{k} \cdot \frac{k^2}{9} = \frac{k}{3} \end{cases} \Rightarrow \begin{cases} x = \frac{k}{3} \\ y = \frac{k}{3} \end{cases}. \quad H(x;y) = \begin{Vmatrix} 6x & -k \\ -k & 6y \end{Vmatrix}.$$

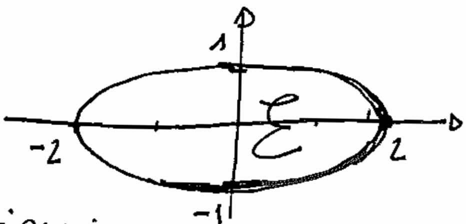
$H(0;0) = \begin{Vmatrix} 0 & -k \\ -k & 0 \end{Vmatrix} \to 0 - k^2 < 0$: Saddle point $\forall k \neq 0$.

If $k = 0$: $f(x;y) = x^3 + y^3$; $f(0;0) = 0$; for $y = 0$ $f(x;0) = x^3 \Rightarrow$ $f(x;0) < 0$ for $x < 0$ and $f(x;0) > 0$ for $x > 0$. So $(0;0)$ is a Saddle point also for $k = 0$.

$$H\left(\frac{k}{3};\frac{k}{3}\right) = \begin{Vmatrix} 2k & -k \\ -k & 2k \end{Vmatrix} \Rightarrow \begin{cases} |H_1| = 2k \\ |H_2| = 4k^2 - k^2 = 3k^2 > 0 \ \forall k \neq 0. \end{cases}$$

So: for $k < 0$: $\begin{cases} |H_1| < 0 \\ |H_2| > 0 \end{cases}$: Maximum Point ; for $k > 0$: $\begin{cases} |H_1| > 0 \\ |H_2| > 0 \end{cases}$: Minimum Point.

II M2) $\begin{cases} \text{Max/min } f(x;y) = x^2 - y^2 \\ \text{s.t.: } x^2 + 4y^2 \leq 4 \end{cases}$



$f(x;y)$ is a continuous function;

$\mathcal{E}$ is a bounded and closed set.

Kuhn-Tucker conditions are satisfied.

$\Lambda = x^2 - y^2 - \lambda(x^2 + 4y^2 - 4)$ $\boxed{MfEA3}$

<u>for $\lambda = 0$</u>

$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = -2y = 0 \\ x^2 + 4y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 + 0 \leq 4 \\ \text{valid} \end{cases}$. $H(x;y) = H(0;0) = \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}$: $|H_2| = -4 < 0$ so $(0;0)$ is a Saddle point.

<u>for $\lambda \neq 0$</u>

$\begin{cases} \Lambda'_x = 2x - 2\lambda x = 2x(1-\lambda) = 0 \\ \Lambda'_y = -2y - 8\lambda y = -2y(1 + 4\lambda) = 0 \\ x^2 + 4y^2 = 4 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \text{Saddle} \end{cases}$ or $\begin{cases} \lambda = 1 \\ y = 0 \\ x^2 = 4 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 0 \\ \lambda = 1 \\ \text{Max?} \end{cases}$ and $\begin{cases} x = -2 \\ y = 0 \\ \lambda = 1 \\ \text{Max?} \end{cases}$

or $\begin{cases} x = 0 \\ \lambda = -\frac{1}{4} \\ 4y^2 = 4 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda = -\frac{1}{4} \\ \text{min?} \end{cases}$ and $\begin{cases} x = 0 \\ y = -1 \\ \lambda = -\frac{1}{4} \\ \text{min?} \end{cases}$ or $\begin{cases} \lambda = 1 \\ \lambda = -\frac{1}{4} \\ \text{no solutions.} \end{cases}$

$f(2;0) = f(-2;0) = 4 \Rightarrow (2;0)$ and $(-2;0)$ are Maximum points.

$f(0;1) = f(0;-1) = -1 \Rightarrow (0;1)$ and $(0;-1)$ are minimum points.

II M3) $f(x;y) = xy - y^2$ differentiable function; $u = (\cos\alpha; \operatorname{sen}\alpha)$.

$D_u f(1;1) = \nabla f(1;1) \circ u = (y; x-2y)\big|_{(1;1)} \cdot (\cos\alpha; \operatorname{sen}\alpha) = (1;-1)(\cos\alpha; \operatorname{sen}\alpha) = \cos\alpha - \operatorname{sen}\alpha.$

$D_u f(1;-1) = \nabla f(1;1) \cdot u = (y; x-2y)\big|_{(1;-1)} \cdot (\cos\alpha; \operatorname{sen}\alpha) = (-1;3)(\cos\alpha; \operatorname{sen}\alpha) = 3\operatorname{sen}\alpha - \cos\alpha.$

$\begin{cases} D_u f(1;1) = \cos\alpha - \operatorname{sen}\alpha = 0 \\ D_u f(1;-1) = 3\operatorname{sen}\alpha - \cos\alpha = \sqrt{2} \end{cases} \Rightarrow \begin{cases} \cos\alpha = \operatorname{sen}\alpha \\ 3\operatorname{sen}\alpha - \operatorname{sen}\alpha = 2\operatorname{sen}\alpha = \sqrt{2} \end{cases} \Rightarrow \begin{cases} \cos\alpha = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \\ \operatorname{sen}\alpha = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \end{cases}$

So $\alpha = \frac{\pi}{4} \Rightarrow u = \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$.

II M4) $\begin{cases} f(x;y;z) = \operatorname{sen}(x-y) + \cos(z-2x) = 1 \\ g(x;y;z) = e^{2x-z} - e^{y-x} = 0 \end{cases}$; $\begin{cases} f(1;1;2) = \operatorname{sen} 0 + \cos 0 = 0 + 1 = 1 \\ g(1;1;2) = e^0 - e^0 = 1 - 1 = 0. \end{cases}$

$\frac{\partial(f;g)}{\partial(x;y;z)} = \begin{Vmatrix} \cos(x-y) + 2\operatorname{sen}(z-2x) & -\cos(x-y) & -\operatorname{sen}(z-2x) \\ 2e^{2x-z} + e^{y-x} & -e^{y-x} & -e^{2x-z} \end{Vmatrix}$

$\frac{\partial(f;g)}{\partial(x;y;z)}(1;1;2) = \begin{Vmatrix} \cos 0 + 2\operatorname{sen} 0 & -\cos 0 & -\operatorname{sen} 0 \\ 2e^0 + e^0 & -e^0 & -e^0 \end{Vmatrix} = \begin{Vmatrix} 1 & -1 & 0 \\ 3 & -1 & -1 \end{Vmatrix}.$

Since $\begin{vmatrix} -1 & 0 \\ -1 & -1 \end{vmatrix} = 1 \neq 0$ it is possible to define an implicit function $x \to (y(x); z(x))$.

MfEA4

$$\frac{dy}{dx}(1) = - \frac{\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 0 \\ -1 & -1 \end{vmatrix}} = - \frac{-1}{1} = 1;$$

$$\frac{dz}{dx}(1) = - \frac{\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix}}{\begin{vmatrix} -1 & 0 \\ -1 & -1 \end{vmatrix}} = - \frac{-3+1}{1} = 2.$$

Equation of the tangent line at $x = 1$:

$$t(x) = (1;2) + x \cdot (1;2) = (1+x; 2+2x) = (y_0; z_0) + x \cdot (y'_0; z'_0).$$