Task Mathematics for Economic Applications 6/9/2013 $\boxed{\text{MfEA1}}$

IM1) $z_1\cdot z_2 = 2\cdot 4\cdot\left(\cos\frac{\pi}{5}+i\,\text{sen}\frac{\pi}{5}\right)\left(\cos\frac{4}{5}\pi+i\,\text{sen}\frac{4}{5}\pi\right)=8\cdot\left(\cos\left(\frac{\pi}{5}+\frac{4}{5}\pi\right)+i\,\text{sen}\left(\frac{\pi}{5}+\frac{4}{5}\pi\right)\right)=$

$= 8\left(\cos\pi + i\,\text{sen}\,\pi\right) = -8.$

$\sqrt[3]{-8} = \sqrt[3]{8}\cdot\left(\cos\left(\frac{\pi}{3}+k\cdot\frac{2\pi}{3}\right)+i\,\text{sen}\left(\frac{\pi}{3}+k\frac{2\pi}{3}\right)\right);\ 0\le k\le 2.$

for $k=0$: $2\left(\cos\frac{\pi}{3}+i\,\text{sen}\frac{\pi}{3}\right)=2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=1+i\sqrt{3};$

for $k=1$: $2\left(\cos\pi+i\,\text{sen}\,\pi\right)=2(-1)=-2;$

for $k=2$: $2\left(\cos\frac{5}{3}\pi+i\,\text{sen}\frac{5}{3}\pi\right)=2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=1-i\sqrt{3}.$

IM2) $\begin{Vmatrix}0&1&1\\1&-1&1\\-1&0&1\end{Vmatrix}\cdot\begin{Vmatrix}x\\y\\z\end{Vmatrix}=\begin{Vmatrix}1\\-2\\2\end{Vmatrix}\Rightarrow\begin{Vmatrix}x\\y\\z\end{Vmatrix}=\begin{Vmatrix}0&1&1\\1&-1&1\\-1&0&1\end{Vmatrix}^{-1}\cdot\begin{Vmatrix}1\\-2\\2\end{Vmatrix}.$

$\begin{vmatrix}0&1&1\\1&-1&1\\-1&0&1\end{vmatrix}=\begin{vmatrix}0&0&1\\1&-2&1\\-1&-1&1\end{vmatrix}=1\cdot\begin{vmatrix}1&-2\\-1&-1\end{vmatrix}=1(-1-2)=-3.$

$\begin{Vmatrix}0&1&1\\1&-1&1\\-1&0&1\end{Vmatrix}\xrightarrow{\text{ADJ}}\begin{Vmatrix}-1&-2&-1\\-1&1&-1\\2&1&-1\end{Vmatrix}\xrightarrow{T}\begin{Vmatrix}-1&-1&2\\-2&1&1\\-1&-1&-1\end{Vmatrix}\xrightarrow{\frac{1}{|A|}}\begin{Vmatrix}\frac{1}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{Vmatrix}.$

$\begin{Vmatrix}x\\y\\z\end{Vmatrix}=\begin{Vmatrix}\frac{1}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{2}{3}&-\frac{1}{3}&-\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{Vmatrix}\cdot\begin{Vmatrix}1\\-2\\2\end{Vmatrix}=\begin{Vmatrix}\frac{1}{3}-\frac{2}{3}-\frac{4}{3}\\\frac{2}{3}+\frac{2}{3}-\frac{2}{3}\\\frac{1}{3}-\frac{2}{3}+\frac{2}{3}\end{Vmatrix}=\begin{Vmatrix}-\frac{5}{3}\\\frac{2}{3}\\\frac{1}{3}\end{Vmatrix}.$

IM3) $\begin{Vmatrix}0&1&1\\k&0&k\\0&1&1\end{Vmatrix}\to\begin{vmatrix}-\lambda&1&1\\k&-\lambda&k\\0&1&1-\lambda\end{vmatrix}=\begin{vmatrix}-\lambda&0&1\\k&-\lambda-k&k\\0&\lambda&1-\lambda\end{vmatrix}=(-\lambda)\left((-\lambda-k)(1-\lambda)-\lambda k\right)+1\cdot(k\lambda-0)=$

$=(-\lambda)\left(\lambda^2-\lambda+k\lambda-k-k\lambda\right)+k\lambda=(-\lambda)\left(\lambda^2-\lambda-k-k\right)=(-\lambda)\left(\lambda^2-\lambda-2k\right)=0.$

$\lambda=0$ fixed eigenvalue.

•) $\lambda^2-\lambda-2k=0$ for $\lambda=0$ if $0-0-2k=0\Rightarrow k=0.$

For $k=0$: $\lambda^2-\lambda=\lambda(\lambda-1)=0\Rightarrow\lambda_1=\lambda_2=0;\ \lambda_3=1.$

For $k=0$ and $\lambda=0$: $\|A-0\,\mathbb{I}\|=\begin{Vmatrix}0&1&1\\0&0&0\\0&1&1\end{Vmatrix}$: RANK$(A-0\,\mathbb{I})=1\Rightarrow m_0^g=3-1=2=m_0^a$

So the matrix for $k=0$ is a diagonalizable one.

••) $\lambda^2-\lambda-2k=0\Rightarrow\lambda=\dfrac{1\pm\sqrt{1+8k}}{2}\Rightarrow\Delta=0$ for $k=-\frac{1}{8}.$

For $k=-\frac{1}{8}$: $\lambda_1=0;\ \lambda_2=\lambda_3=\frac{1}{2}.$

For $k=-\frac{1}{8}$ and $\lambda=\frac{1}{2}$: $\|A-\frac{1}{2}\mathbb{I}\|=\begin{Vmatrix}-\frac{1}{2}&1&1\\-\frac{1}{8}&-\frac{1}{2}&-\frac{1}{8}\\0&1&\frac{1}{2}\end{Vmatrix}\to\begin{Vmatrix}-1&2&2\\8&4&1\\0&2&1\end{Vmatrix}$; As $\begin{vmatrix}8&4\\0&2\end{vmatrix}=16\ne0$

RANK$\left(A-\frac{1}{2}\mathbb{I}\right)=2\Rightarrow m_{\frac{1}{2}}^g=3-2=1<2=m_{\frac{1}{2}}^a$ and the matrix is not a diagonalizable one.

For every $k \neq 0$ and $k \neq -\frac{1}{8}$ we have distinct eigenvalues, and so the matrix is a diagonalizable one. $\boxed{\text{MfEA2}}$

IM4) $g(f(x)) = \begin{Vmatrix} 1 & k & 1 \\ 0 & 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & k \\ 1 & -1 \\ k & 0 \end{Vmatrix} = \begin{Vmatrix} 1+k+k & k-k+0 \\ 0+1+k & 0-1+0 \end{Vmatrix} = \begin{Vmatrix} 1+2k & 0 \\ 1+k & -1 \end{Vmatrix}$.

Dim (Imm) = Dim (Ker) iff $\begin{vmatrix} 1+2k & 0 \\ 1+k & -1 \end{vmatrix} = 0 \Rightarrow -1-2k = 0 \Rightarrow k = -\frac{1}{2}$.

For $k = -\frac{1}{2}$ $\text{RANK}(B \cdot A) = 1 \Rightarrow$ Dim (Imm) $= 1$ and Dim (Ker) $= 2-1 = 1$.

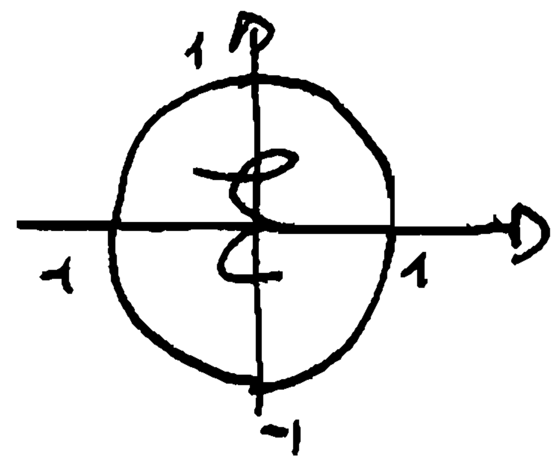
IIM1) $f(x;y) = x^2 - xy + ky^2$. f is a twice differentiable function.

$$\begin{cases} f'_x = 2x - y = 0 \\ f'_y = -x + 2ky = 0 \end{cases} \Rightarrow \begin{cases} y = 2x \\ -x + 4kx = 0 \end{cases} \Rightarrow \begin{cases} y = 2x \\ x(4k-1) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}.$$

$$H = \begin{Vmatrix} 2 & -1 \\ -1 & 2k \end{Vmatrix} = H(0;0) \Rightarrow \begin{cases} \begin{cases} 2k > 0 \\ 4x - 1 > 0 \end{cases} \Rightarrow \begin{cases} k > 0 \\ k > \frac{1}{4} \end{cases} \Rightarrow k > \frac{1}{4} : \text{Minimum point;} \\ 4x - 1 < 0 \Rightarrow \text{Saddle Point.} \end{cases}$$

For $k = \frac{1}{4}$: $x^2 - xy + \frac{1}{4}y^2 = \left(x - \frac{1}{2}y\right)^2 \Rightarrow$ all the points of the straight line having equation $y = 2x$ are minimum point.

IIM2) $\begin{cases} \text{Max/min } f(x;y) = x^2 - xy + y^2 \\ \text{s.t. } x^2 + y^2 \leq 1 \end{cases}$



$\mathcal{E}$ is a bounded and closed set. $f(x;y)$ is a continuous function, constraint is a qualified one.

$$\Lambda = x^2 - xy + y^2 - \lambda(x^2 + y^2 - 1)$$

For $\lambda = 0$

$$\begin{cases} \Lambda'_x = 2x - y = 0 \\ \Lambda'_y = -x + 2y = 0 \\ x^2 + y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \leq 1 \end{cases} ; \; H = \begin{Vmatrix} 2 & -1 \\ -1 & 2 \end{Vmatrix} = H(0;0) \Rightarrow \begin{cases} 2 > 0 \\ 4 - 1 = 3 > 0 \end{cases} : \text{Minimum point.}$$

For $\lambda \neq 0$

$$\begin{cases} \Lambda'_x = 2x - y - 2\lambda x = 0 \\ \Lambda'_y = -x + 2y - 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} \overbrace{x + y - 2\lambda x - 2\lambda y}^{\mathcal{E}_1 + \mathcal{E}_2} = x + y - 2\lambda(x+y) = (x+y)(1-2\lambda) = 0 \\ -x + 2y - 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y = -x \\ -x - 2x + 2\lambda x = -3x + 2\lambda x = 0 \Rightarrow \lambda = \frac{3}{2} \\ 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \\ \lambda = \frac{3}{2} > 0 \end{cases} \text{ and } \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = \frac{1}{\sqrt{2}} \\ \lambda = \frac{3}{2} > 0 \end{cases}$$

Max ? Max ?

MfEA3

$$\begin{cases} \lambda = \frac{1}{2} \\ -x+2y-y = -x+y = 0 \Rightarrow y = x \\ 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \frac{1}{\sqrt{2}} \\ \lambda = \frac{1}{2} > 0 \end{cases} \text{ and } \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \\ \lambda = \frac{1}{2} > 0 \end{cases}$$

Max ? Max ?

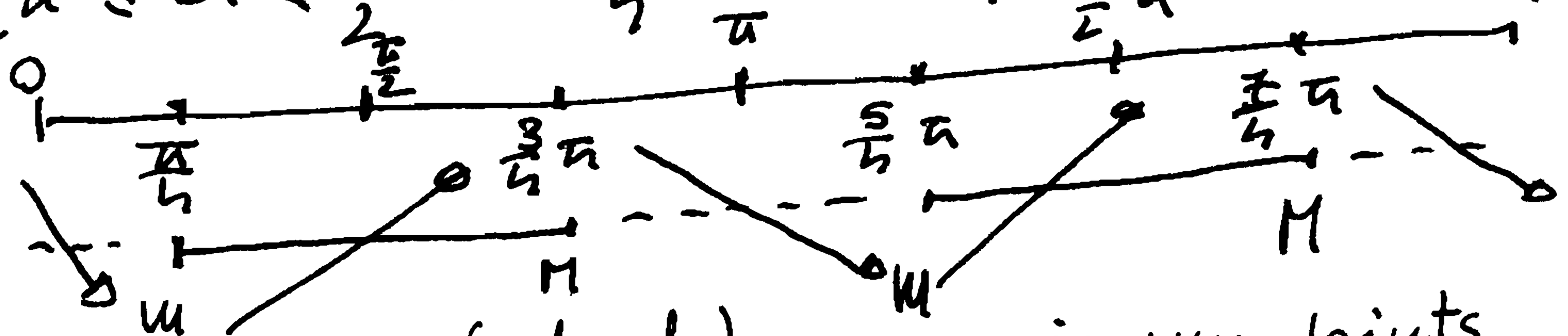
$$f\left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = \frac{3}{2}; \; f\left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}.$$

We study the objective function on the boundary, posing $(x;y) = (\cos t; \operatorname{sen} t)$

$f(t) = \cos^2 t - \operatorname{sen} t \cos t + \operatorname{sen}^2 t = 1 - \operatorname{sen}^2 t - \operatorname{sen} t \cos t + \operatorname{sen}^2 t = 1 - \operatorname{sen} t \cos t = 1 - \frac{1}{2}\operatorname{sen} 2t.$

$f'(t) = -\cos 2t \geq 0$ if $\cos 2t \leq 0 \Rightarrow \frac{\pi}{2} \leq 2t \leq \frac{3}{2}\pi \Rightarrow \frac{\pi}{4} \leq t \leq \frac{3}{4}\pi$ and

$\frac{5}{2}\pi \leq 2t \leq \frac{7}{2}\pi \Rightarrow \frac{5}{4}\pi \leq t \leq \frac{7}{4}\pi$



So $\left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$ are maximum points.

$\left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right)$ are saddle (inflection) points.

II M3) $D^2_{u,v} f(1;-1;1) = u \cdot H(1;-1;1) \cdot v^T$. $f(x;y;z) = x^2 + yz$

$\nabla f = (2x; z; y)$; $H(f) = \begin{Vmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} = H(1;-1;1)$. $u = \left(\frac{1}{\sqrt{2}}; 0; \frac{1}{\sqrt{2}}\right)$; $v = \left(0; \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$.

$$D^2_{u,v} f(1;-1;1) = \begin{Vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Vmatrix} = \begin{Vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Vmatrix} = \frac{1}{2}.$$

II M4) $f(x;y) = x^3 - y^3 - x^2 + y^2 = 0$ $\quad \nabla f = (3x^2 - 2x; -3y^2 + 2y)$.

$\nabla f(0;0) = (0;0)$: it is not posible to define an implicit function.

$\nabla f(1;1) = (1;-1)$: $f'_y \neq 0$: it is posible to define $y = y(x)$.

$$y' = -\frac{f'_x}{f'_y} \Rightarrow y'(1) = -\frac{1}{-1} = 1.$$

$$H = \begin{Vmatrix} 6x-2 & 0 \\ 0 & 2-6y \end{Vmatrix}; \; H(1;1) = \begin{Vmatrix} 4 & 0 \\ 0 & -4 \end{Vmatrix}$$

$$y'' = -\frac{f''_{xx} + 2f''_{xy} \cdot y' + f''_{yy} \cdot (y')^2}{f'_y}; \; y''(1) = -\frac{4 + 2 \cdot 0 \cdot 1 + (-4) \cdot (1^2)}{-1} = \frac{4-4}{1} = 0.$$