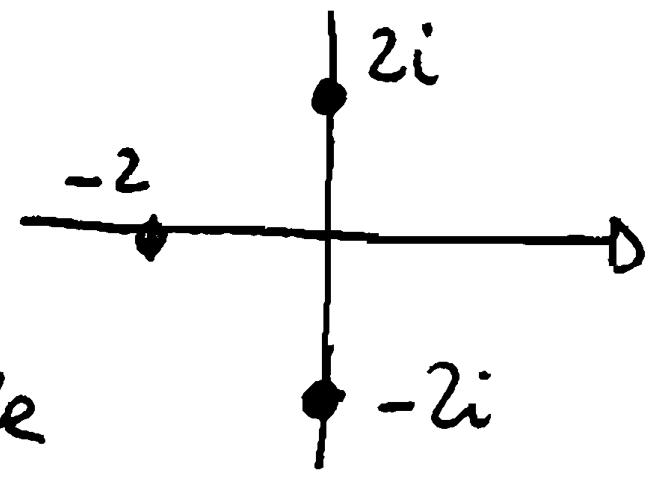
# Task Mathematics for Economic Applications 23/9/2013 $\boxed{\text{MfEA1}}$

**IM1)** $x^3+2x^2+4x+8=0$ if $x=-2$: $-8+8-8+8=0$

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 4 | 8 |
| $-2$ | | $-2$ | 0 | $-8$ |
| | 1 | 0 | 4 | 0 |

$x^3+2x^2+4x+8=(x+2)(x^2+4)=0 \Rightarrow$

$\Rightarrow x_1=-2;\ x_2=2i;\ x_3=-2i.$




Representing the points in the complex plane we get:
so they are not the vertices of an equilateral triangle inscribed in a circle with center at the origin $\Rightarrow$ they are not the three third order roots of a complex number.

**IM2)** $A\cdot P=P\cdot B \Rightarrow \left\|\begin{matrix}2&1\\1&2\end{matrix}\right\|\cdot\left\|\begin{matrix}x_1&y_1\\x_2&y_2\end{matrix}\right\|=\left\|\begin{matrix}x_1&y_1\\x_2&y_2\end{matrix}\right\|\cdot\left\|\begin{matrix}-3&-3\\8&7\end{matrix}\right\| \Rightarrow \begin{cases}2x_1+x_2=-3x_1+8y_1\\2y_1+y_2=-3x_1+7y_1\\x_1+2x_2=-3x_2+8y_2\\y_1+2y_2=-3x_2+7y_2\end{cases}\Rightarrow$

$$\Rightarrow\begin{cases}5x_1+x_2-8y_1+0\cdot y_2=0\\3x_1+0\cdot x_2-5y_1+y_2=0\\x_1+5x_2+0y_1-8y_2=0\\0\cdot x_1+3x_2+y_1-5y_2=0\end{cases}\Rightarrow\left\|\begin{matrix}5&1&-8&0\\3&0&-5&1\\1&5&0&-8\\0&3&1&-5\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}1&5&0&-8\\0&3&1&-5\\5&1&-8&0\\3&0&-5&1\end{matrix}\right\|\Rightarrow$$

$$\Rightarrow\left\|\begin{matrix}1&5&0&-8\\0&3&1&-5\\0&-24&-8&40\\0&-15&-5&25\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}1&5&0&-8\\0&3&1&-5\\0&3&1&-5\\0&3&1&-5\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}1&5&0&-8\\0&3&1&-5\\0&0&0&0\\0&0&0&0\end{matrix}\right\|$$

$R_3\leftarrow R_3-5R_1$; $R_4\leftarrow R_4-3R_1$ $\qquad$ $R_3\leftarrow -\frac{1}{8}R_3$; $R_4\leftarrow -\frac{1}{5}R_4$ $\qquad$ $R_3\leftarrow R_3-R_2$; $R_4\leftarrow R_4-R_2$

So the RANK $=2$ and the system has $\infty^2$ solutions.

A and B are similar matrices.

**IM3)** $\begin{cases}x_1-2x_2+2x_3+x_4=-1\\2x_1+3x_2+x_3+x_4=2\\x_1-9x_2+mx_3+Kx_4=h\end{cases}\Rightarrow\left\|\begin{matrix}1&-2&2&1&\vdots&-1\\2&3&1&1&\vdots&2\\1&-9&m&K&\vdots&h\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}1&-2&2&1&\vdots&-1\\0&7&-3&-1&\vdots&4\\0&-7&m-2&K-1&\vdots&h+1\end{matrix}\right\|\Rightarrow$

$R_2\leftarrow R_2-2R_1$; $R_3\leftarrow R_3-R_1$

$$\Rightarrow\left\|\begin{matrix}1&-2&2&1&\vdots&-1\\0&7&-3&-1&\vdots&4\\0&0&m-5&K-2&\vdots&h+5\end{matrix}\right\|.$$

$R_3\leftarrow R_3+R_2$

If $m=5$ and $K=2$ and $h=-5$: RANK$(A)=2=$ RANK$(A|Y)\Rightarrow\infty^2$ Solutions

If $m\neq5$ or $K\neq2$ $\forall h$: RANK$(A)=3=$ RANK$(A|Y)\Rightarrow\infty^1$ Solutions

If $m=5$ and $K=2$ but $h\neq-5$: RANK$(A)=2<3=$ RANK$(A|Y)\Rightarrow$ no solutions.

**IM4)** $A\cdot X_1=\underline{0}$; $A\cdot X_2=\underline{0}$; $A\cdot X_3=Y_1$. $A=\left\|\begin{matrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{matrix}\right\|.$

MfEA2

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 0 \\ 0 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases};$$

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix} \cdot \begin{Vmatrix} 0 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} b_1 + c_1 = 0 \\ b_2 + c_2 = 0 \\ b_3 + c_3 = 0 \end{cases} \Rightarrow \begin{cases} b_1 = -c_1 \\ b_2 = -c_2 \\ b_3 = -c_3 \end{cases} \Rightarrow A = \begin{Vmatrix} 0 & -c_1 & c_1 \\ 0 & -c_2 & c_2 \\ 0 & -c_3 & c_3 \end{Vmatrix}.$$

$$\begin{Vmatrix} 0 & -c_1 & c_1 \\ 0 & -c_2 & c_2 \\ 0 & -c_3 & c_3 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 2 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 3 \\ -3 \\ -6 \end{Vmatrix} \Rightarrow \begin{cases} -2c_1 - c_1 = -3c_1 = 3 \Rightarrow c_1 = -1 \Rightarrow b_1 = 1 \\ -2c_2 - c_2 = -3c_2 = -3 \Rightarrow c_2 = 1 \Rightarrow b_2 = -1. \\ -2c_3 - c_3 = -3c_3 = -6 \Rightarrow c_3 = 2 \Rightarrow b_3 = -2 \end{cases}$$

$$A = \begin{Vmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{Vmatrix} \to \begin{vmatrix} -\lambda & 1 & -1 \\ 0 & -1-\lambda & 1 \\ 0 & -2 & 2-\lambda \end{vmatrix} = (-\lambda)\Big((-1-\lambda)(2-\lambda)+2\Big) = -\lambda(\lambda^2-\lambda-2+2) = -\lambda^2(\lambda-1) = 0$$

$\lambda_1 = \lambda_2 = 0;\ \lambda_3 = 1.$

<u>For $\lambda = 0$</u> $\quad \|A - 0\cdot I\| = \begin{Vmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{Vmatrix} \to \begin{Vmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$: RANK $(A - 0I) = 1 \Rightarrow m_0^g = 3 - 1 = m_0^a$

so the matrix is a diagonalitable one.

II M1) $f(x;y) = \begin{cases} \dfrac{x^2}{x+y} & : x \neq -y \\ 0 & : x = -y \end{cases}$. $\displaystyle D_v f(0;0) = \lim_{t\to 0} \frac{f((0;0)+t(\cos\alpha;\operatorname{sen}\alpha)) - f(0;0)}{t} =$

$$= \lim_{t\to 0}\left(\frac{t^2\cos^2\alpha}{t\cos\alpha + t\operatorname{sen}\alpha} - 0\right)\cdot\frac{1}{t} = \lim_{t\to 0}\frac{t^2\cos^2\alpha}{t^2(\cos\alpha+\operatorname{sen}\alpha)} = \frac{\cos^2\alpha}{\cos\alpha+\operatorname{sen}\alpha}.$$

The derivative exists if $\cos\alpha + \operatorname{sen}\alpha \neq 0 \Rightarrow \cos\alpha \neq -\operatorname{sen}\alpha \Rightarrow x \neq y$.

If $\cos\alpha = -\operatorname{sen}\alpha \iff x = -y$ : $\displaystyle\lim_{t\to 0}\frac{f((0;0)+tv) - f(0;0)}{t} = \lim_{t\to 0}\frac{0}{t} = 0.$

The function has directional derivative $\forall v$.

II M2) $\begin{cases} f(x;y;z) = xyz - e^{x-y} = 0 \\ g(x;y;z) = xyz - e^{y-z} = 0 \end{cases}$; $\begin{cases} f(1;1;1) = 0 \\ g(1;1;1) = 0 \end{cases}$.

$$\frac{\partial(f;g)}{\partial(x;y;z)} = \begin{Vmatrix} yz - e^{x-y} & xz + e^{x-y} & xy \\ yz & xz - e^{y-z} & xy + e^{y-z} \end{Vmatrix};\ \frac{\partial(f;g)}{\partial(x;y;z)}(1;1;1) = \begin{Vmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \end{Vmatrix}.$$

Since $\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 4 \neq 0$ the implicit function $x \to (y(x); z(x))$ exists.

$$\frac{dy}{dx} = -\frac{\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}} = -\frac{-1}{4} = \frac{1}{4};\ \frac{dz}{dx} = -\frac{\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}} = -\frac{2}{4} = -\frac{1}{2}.$$

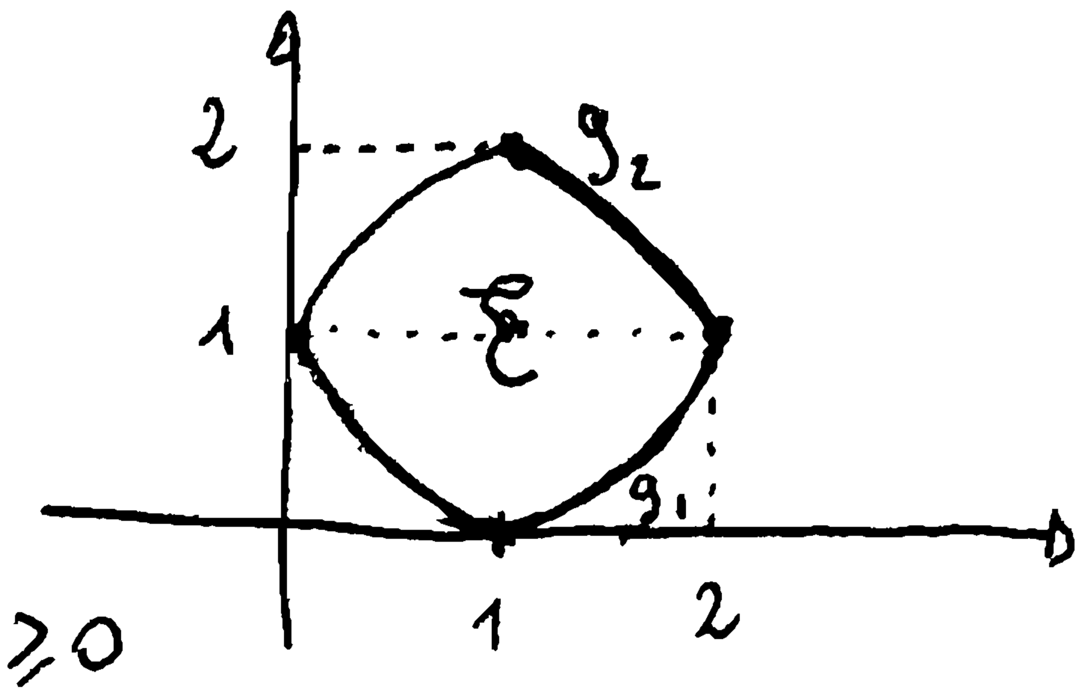
Equation of the tangent line at $x = 1$: $\quad (1;1) + x\cdot\left(\frac{1}{4}; -\frac{1}{2}\right) \Rightarrow x \to \left(1 + \frac{1}{4}x; 1 - \frac{1}{2}x\right).$

$\boxed{\text{MfEA3}}$

**II M3)** $\begin{cases} \text{Max/min } f(x;y) = x - y \\ \text{s.t. } \begin{cases} x^2 - 2x - y + 1 \leq 0 \\ x^2 - 2x + y - 1 \leq 0 \end{cases} \end{cases}$

$g_1: y = x^2 - 2x + 1 = (x-1)^2$

$g_2: y = -x^2 + 2x + 1 = -(x-1)^2 + 2$



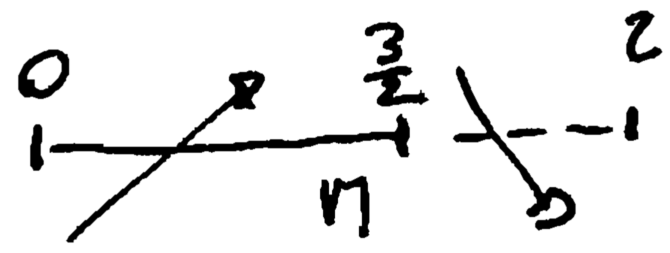

$(x-1)^2 = 2 - (x-1)^2 \Rightarrow (x-1)^2 = 1 \Rightarrow x - 1 = \pm 1 \Rightarrow$

$\Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases}$ and $\begin{cases} x = 0 \\ y = 1 \end{cases}$.

$\nabla f = (1;-1) \Rightarrow$ no free stationary points.

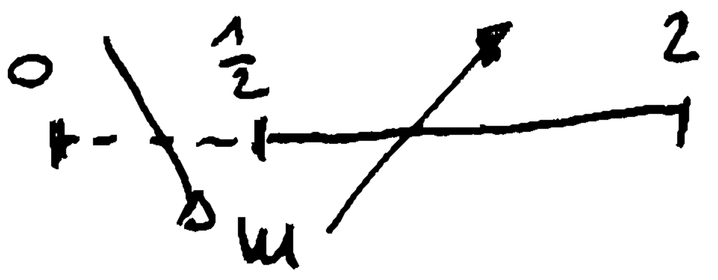
If $y = (x-1)^2 \Rightarrow f(x) = x - (x-1)^2 \Rightarrow f'(x) = 1 - 2(x-1) \geq 0$

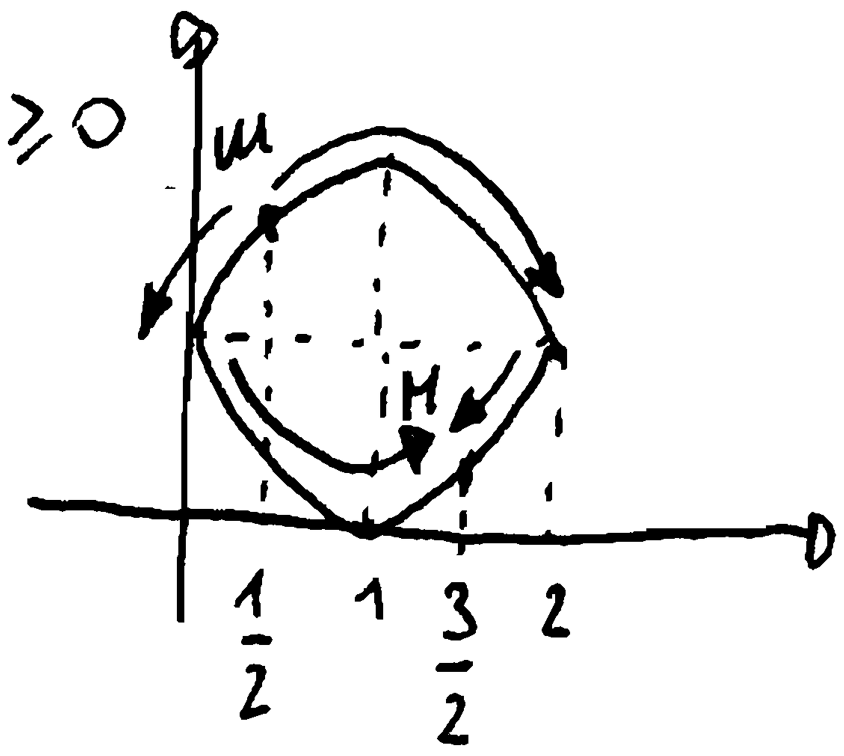
$\Rightarrow 2x \leq 3 \Rightarrow x \leq \frac{3}{2}$




If $y = 2 - (x-1)^2 \Rightarrow f(x) = x - 2 + (x-1)^2 \Rightarrow f'(x) = 1 + 2(x-1) \geq 0$

$\Rightarrow 2x \geq 1 \Rightarrow x \geq \frac{1}{2}$




At the point $x = \frac{1}{2}$; $y = \frac{7}{4}$; $f(\frac{1}{2};\frac{7}{4}) = -\frac{5}{4}$: absolute minimum

At the point $x = \frac{3}{2}$; $y = \frac{1}{4}$; $f(\frac{3}{2};\frac{1}{4}) = \frac{5}{4}$: absolute maximum.

**II M4)** $\begin{cases} \text{Max/min } f(x;y;z) = xy - z \\ \text{s.t.: } xz + yz = 4 \end{cases}$ $\quad \bullet\ \Lambda = xy - z - \lambda(xz + yz - 4)$

$$\begin{cases} \Lambda'_x = y - \lambda z = 0 \\ \Lambda'_y = x - \lambda z = 0 \\ \Lambda'_z = -1 - \lambda x - \lambda y = 0 \\ xz + yz = 4 \end{cases} \Rightarrow \begin{cases} y = \lambda z \\ x = \lambda z \quad \Rightarrow y = x \\ 2\lambda x = -1 \Rightarrow x = -\frac{1}{2\lambda} \\ xz + yz = 4 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2\lambda} \\ y = -\frac{1}{2\lambda} \\ z = \frac{1}{\lambda} \cdot x = -\frac{1}{2\lambda^2} \\ -\frac{1}{2\lambda} \cdot \left(-\frac{1}{2\lambda^2}\right) + \left(-\frac{1}{2\lambda}\right)\left(-\frac{1}{2\lambda^2}\right) = 4 \end{cases} \Rightarrow$$

$$\Rightarrow \frac{2}{4\lambda^3} = 4 \Rightarrow \lambda^3 = \frac{1}{8} \Rightarrow \lambda = \frac{1}{2} \Rightarrow \begin{cases} x = -1 \\ y = -1 \\ z = -2 \\ \lambda = \frac{1}{2} \end{cases}$$

$$\bar{H} = \begin{Vmatrix} 0 & z & z & x+y \\ z & 0 & 1 & -\lambda \\ z & 1 & 0 & -\lambda \\ x+y & -\lambda & -\lambda & 0 \end{Vmatrix}$$

$$\left|\bar{H}_3\left(-1;-1;-2;\tfrac{1}{2}\right)\right| = \begin{vmatrix} 0 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2 & -2 \\ 0 & -1 & 1 \\ -2 & 1 & 0 \end{vmatrix} = (-2)\begin{vmatrix} -2 & -2 \\ -1 & 1 \end{vmatrix} = (-2)(-4) = 8 > 0. \text{ (MAX ??)}$$

$$\left|\bar{H}_4\left(-1;-1;-2;\tfrac{1}{2}\right)\right| = \begin{vmatrix} 0 & -2 & -2 & -2 \\ -2 & 0 & 1 & -\frac{1}{2} \\ -2 & 1 & 0 & -\frac{1}{2} \\ -2 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2 & -2 & -2 \\ 0 & -1 & 1 & 0 \\ -2 & 1 & 0 & -\frac{1}{2} \\ 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = (-2)\begin{vmatrix} -2 & -2 & -2 \\ -1 & 1 & 0 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = (-2)\begin{vmatrix} -4 & -2 & -2 \\ 0 & 1 & 0 \\ -2 & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} =$$

$$= (-2)(1)\begin{vmatrix} -4 & -2 \\ -2 & \frac{1}{2} \end{vmatrix} = (-2)\cdot(-2-4) = 12 > 0$$ : The point $(-1;-1;-2)$ is a saddle point, so the problem has no solutions.