1) The vector v has norm one while gradient and hessian of f are

$\nabla f = (e^{x-y}, -e^{x-y}); \mathcal{H}f = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ -e^{x-y} & e^{x-y} \end{bmatrix}$. Thus

$\mathcal{D}_{vv}f(O) = v^T \cdot \mathcal{H}f(O) \cdot v =$

$(\cos\alpha, \sin\alpha) \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \cos^2\alpha - 2\cos\alpha\sin\alpha + \sin^2\alpha =$

$1 - \sin 2\alpha$. The condition is satisfied iff $\sin 2\alpha = 0$ and this is true for $\alpha = 0 \vee \alpha = \pi/2 \vee \alpha = \pi \vee \alpha = 3\pi/2$.

2) f is differentiable at point $(0,0)$ iff exist an affine function $T(x,y)$ such that:

$\lim_{(x,y)\to(0,0)} \dfrac{f(x,y) - f(0,0) - T(x,y)}{\sqrt{x^2+y^2}} = 0$. Rewritten the limit in polar coordinates with $T(x,y) = ax + by$, we have

$\lim_{\rho\to 0} \dfrac{\frac{\rho\cos\alpha\rho\sin\alpha|\rho\cos\alpha\rho\sin\alpha|}{\rho^2} - (a\rho\cos\alpha + b\rho\sin\alpha)}{\rho} =$

$\lim_{\rho\to 0} \big(\rho\cos\alpha\sin\alpha|\cos\alpha\sin\alpha| - (a\cos\alpha + b\sin\alpha)\big)$ and the limit is 0 only if $a = b = 0$. To conclude that f is differentiable at point $(0,0)$ we must verify that the result is uniformly on α, for this goal consider

$|\rho\cos\alpha\sin\alpha|\cos\alpha\sin\alpha|| = \rho(\cos\alpha\sin\alpha)^2 \le \rho,\ \forall\alpha$ and if $a = b = 0$ we can conclude that $\forall\epsilon > 0,\ \rho < \epsilon \Rightarrow |\rho\cos\alpha\sin\alpha|\cos\alpha\sin\alpha|| < \epsilon$, convergence is uniformly and f is differentiable at point $(0,0)$.

3) Lagrange function of the problem is

$\mathcal{L}(x,y,z,\lambda) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \lambda(x^2+y^2+z^2-3)$.

$\nabla\mathcal{L} = \big(-\frac{1}{x^2} - 2\lambda x,\ -\frac{1}{y^2} - 2\lambda y,\ -\frac{1}{z^2} - 2\lambda z,\ -(x^2+y^2+z^2-3)\big)$. Put the first order condition we have the sistem:

$$\begin{cases} -\frac{1}{x^2} - 2\lambda x = 0 \\ -\frac{1}{y^2} - 2\lambda y = 0 \\ -\frac{1}{z^2} - 2\lambda z = 0 \\ x^2+y^2+z^2 = 3 \end{cases} \Rightarrow \begin{cases} 1 + 2\lambda x^3 = 0 \\ 1 + 2\lambda y^3 = 0 \\ 1 + 2\lambda z^3 = 0 \\ x^2+y^2+z^2 = 3 \end{cases} \Rightarrow \begin{cases} x = -\sqrt[3]{1/2\lambda} \\ y = -\sqrt[3]{1/2\lambda} \\ z = -\sqrt[3]{1/2\lambda} \\ x^2+y^2+z^2 = 3 \end{cases} \Rightarrow$$

$$\begin{cases} x = -\sqrt[3]{1/2\lambda} \\ y = -\sqrt[3]{1/2\lambda} \\ z = -\sqrt[3]{1/2\lambda} \\ 3\sqrt[3]{1/4\lambda^2} = 3 \end{cases} \Rightarrow \begin{cases} x = \mp 1 \\ y = \mp 1 \\ z = \mp 1 \\ \lambda = \pm 1/2 \end{cases}$$

. Two points satisfied first order conditions $P_1 = (1,1,1,-1/2)$ and $P_2 = (-1,-1,-1,1/2)$.

$$\widetilde{\mathcal{H}} = \begin{bmatrix} 0 & 2x & 2y & 2z \\ 2x & \frac{2}{x^3}-2\lambda & 0 & 0 \\ 2y & 0 & \frac{2}{y^3}-2\lambda & 0 \\ 2z & 0 & 0 & \frac{2}{z^3}-2\lambda \end{bmatrix}$$, becouse there's only one constraint

we must analyse third and fourth principal minor of border hessian:

$$\widetilde{\mathcal{H}}_3 = \begin{vmatrix} 0 & 2x & 2y \\ 2x & \frac{2}{x^3}-2\lambda & 0 \\ 2y & 0 & \frac{2}{y^3}-2\lambda \end{vmatrix} = -2x\begin{vmatrix} 2x & 0 \\ 2y & \frac{2}{y^3}-2\lambda \end{vmatrix} + 2y\begin{vmatrix} 2x & \frac{2}{x^3}-2\lambda \\ 2y & 0 \end{vmatrix} =$$

$$-2x^2\left(\tfrac{2}{y^3}-2\lambda\right)-2y^2\left(\tfrac{2}{x^3}-2\lambda\right);$$

$$\widetilde{\mathcal{H}}_4 = \begin{vmatrix} 0 & 2x & 2y & 2z \\ 2x & \frac{2}{x^3}-2\lambda & 0 & 0 \\ 2y & 0 & \frac{2}{y^3}-2\lambda & 0 \\ 2z & 0 & 0 & \frac{2}{z^3}-2\lambda \end{vmatrix} = -2z\begin{vmatrix} 2x & \frac{2}{x^3}-2\lambda & 0 \\ 2y & 0 & \frac{2}{y^3}-2\lambda \\ 2z & 0 & 0 \end{vmatrix} +$$

$$+\left(\tfrac{2}{z^3}-2\lambda\right)\widetilde{\mathcal{H}}_3 = -2z^2\begin{vmatrix} \frac{2}{x^3}-2\lambda & 0 \\ 0 & \frac{2}{y^3}-2\lambda \end{vmatrix} + \left(\tfrac{2}{z^3}-2\lambda\right)\widetilde{\mathcal{H}}_3 =$$

$$-2z^2\left(\tfrac{2}{x^3}-2\lambda\right)\left(\tfrac{2}{y^3}-2\lambda\right)+\left(\tfrac{2}{z^3}-2\lambda\right)\left[-2x^2\left(\tfrac{2}{y^3}-2\lambda\right)-2y^2\left(\tfrac{2}{x^3}-2\lambda\right)\right] =$$

$$-2x^2\left(\tfrac{2}{y^3}-2\lambda\right)\left(\tfrac{2}{z^3}-2\lambda\right)-2y^2\left(\tfrac{2}{x^3}-2\lambda\right)\left(\tfrac{2}{z^3}-2\lambda\right)-2z^2\left(\tfrac{2}{x^3}-2\lambda\right)\left(\tfrac{2}{y^3}-2\lambda\right).$$

Second order conditions:

$\widetilde{\mathcal{H}}_3(P_1) = -12 < 0$; $\widetilde{\mathcal{H}}_4(P_1) = -54 < 0$. P_1 minimum.

$\widetilde{\mathcal{H}}_3(P_2) = +12 > 0$; $\widetilde{\mathcal{H}}_4(P_2) = -54 < 0$. P_2 maximum.

The proposed problem has minimum equal 3 on point $(1,1,1)$ and maximum equal -3 on point $(-1,-1,-1)$.

4) The condition is satisfied on point P, $1\cdot e^3 = 1\cdot e^3$; rewritten the condition as $x\,e^{2y^2+z^2} - y\,e^{x^2+2z^2} = 0$ we calculate the gradient of $f(x,y,z) = x\,e^{2y^2+z^2} - y\,e^{x^2+2z^2}$.

$\nabla f = (e^{2y^2+z^2} - 2xye^{x^2+2z^2}, 4xye^{2y^2+z^2} - e^{x^2+2z^2}, 2xze^{2y^2+z^2} - 4yz\,e^{x^2+2z^2})$ and $\nabla f(P) = (-e^3, 3\,e^3, -2\,e^3)$, but any element of $\nabla f(P)$ is different from zero thus we can have in implicy form a function of kind $(x,y)\mapsto z(x,y)$ or $(x,z)\mapsto y(x,z)$ or $(y,z)\mapsto x(y,z)$. If we choose the first kind, $\nabla z(1,1) = \left(-\frac{f'_x(P)}{f'_z(P)}, -\frac{f'_y(P)}{f'_z(P)}\right) = \left(-\frac{-e^3}{-2\,e^3}, -\frac{3\,e^3}{-2\,e^3}\right) = \left(-\frac{1}{2}, \frac{3}{2}\right)$. The equation of tangent plane will be

$$z-1 = \nabla z(1,1)\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \Leftrightarrow z-1 = -\tfrac{1}{2}(x-1)+\tfrac{3}{2}(y-1) \Leftrightarrow$$

$$x - 3\,y + 2\,z = 0.$$

5) $\nabla z = (3x^2 + \alpha y, \alpha x + 3y^2)$, $\mathcal{H}z = \begin{bmatrix} 6x & \alpha \\ \alpha & 6y \end{bmatrix}$. $z''_{xx}(1,0) = z''_{yy}(0,1) = 6$, $z''_{xy}(1,0) = z''_{yx}(0,1) = \alpha$; the condition is true when $36 = \alpha^2 \Rightarrow \alpha = \pm 6$.