Intermediate test Mathematics for Economic Applications 9/12/13 | MfEA1

IM1) $z^4 + iz = z(z^3+i) = 0 \Rightarrow z = 0$ and $z^3 = -i \Rightarrow z = \sqrt[3]{-i}$.

$-i = 1\cdot\left(\cos\frac{3}{2}\pi + i\,\mathrm{sen}\,\frac{3}{2}\pi\right)$.

$\sqrt[3]{-i} = \sqrt[3]{1}\cdot\left(\cos\left(\frac{\pi}{2} + k\cdot\frac{2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi}{2} + k\cdot\frac{2\pi}{3}\right)\right);\ 0 \le k \le 2.$

for $k=0$: $1\cdot\left(\cos\frac{\pi}{2} + i\,\mathrm{sen}\,\frac{\pi}{2}\right) = 1\cdot(0+i) = i$;

for $k=1$: $1\cdot\left(\cos\left(\frac{\pi}{2}+\frac{2}{3}\pi\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}+\frac{2}{3}\pi\right)\right) = \cos\frac{7}{6}\pi + i\,\mathrm{sen}\,\frac{7}{6}\pi = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$;

for $k=2$: $1\cdot\left(\cos\left(\frac{\pi}{2}+\frac{4}{3}\pi\right) + i\,\mathrm{sen}\left(\frac{\pi}{2}+\frac{4}{3}\pi\right)\right) = \cos\frac{11}{6}\pi + i\,\mathrm{sen}\,\frac{11}{6}\pi = \frac{\sqrt{3}}{2} - \frac{1}{2}i$.

IM2) $\left\|\begin{array}{cccc:c} 1 & -2 & 2 & -3 & y_1 \\ 2 & 0 & 2 & 1 & y_2 \\ 4 & 4 & 2 & 9 & y_3 \end{array}\right\| \longrightarrow \left\|\begin{array}{cccc:c} 1 & -2 & 2 & -3 & y_1 \\ 0 & 4 & -2 & 7 & y_2 - 2y_1 \\ 0 & 12 & -6 & 21 & y_3 - 4y_1 \end{array}\right\| \longrightarrow$

$R_2 \leftarrow R_2 - 2R_1$; $R_3 \leftarrow R_3 - 4R_1$

$\longrightarrow \left\|\begin{array}{cccc:c} 1 & -2 & 2 & -3 & y_1 \\ 0 & 4 & -2 & 7 & y_2 - 2y_1 \\ 0 & 0 & 0 & 0 & y_3 - 4y_1 - 3(y_2 - 2y_1) \end{array}\right\|$. From Rouché-Capelli Theorem follows:

$R_3 \leftarrow R_3 - 3R_2$

$y_3 - 4y_1 - 3y_2 + 6y_1 = y_3 + 2y_1 - 3y_2 = 0 \Rightarrow y_3 = 3y_2 - 2y_1$.

$Y = (y_1;\ y_2;\ 3y_2 - 2y_1)$.

IM3) $X = \left\|\begin{matrix} 1 & 2 \\ 1 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ -2 \end{matrix}\right\| = \left\|\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}\right\| \cdot \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\| \Rightarrow$

$\Rightarrow \left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\| = \left\|\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}\right\|^{-1} \cdot \left\|\begin{matrix} 1 & 2 \\ 1 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ -2 \end{matrix}\right\|$.

$\left|\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}\right| = 4 - 3 = 1$; $\left\|\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}\right\| \xrightarrow{Adj} \left\|\begin{matrix} 2 & -1 \\ -3 & 2 \end{matrix}\right\| \xrightarrow{T} \left\|\begin{matrix} 2 & -3 \\ -1 & 2 \end{matrix}\right\| = A^{-1}$.

$\left\|\begin{matrix} x_1 \\ x_2 \end{matrix}\right\| = \left\|\begin{matrix} 2 & -3 \\ -1 & 2 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 & 2 \\ 1 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ -2 \end{matrix}\right\| = \left\|\begin{matrix} 2 & -3 \\ -1 & 2 \end{matrix}\right\| \cdot \left\|\begin{matrix} -3 \\ -1 \end{matrix}\right\| = \left\|\begin{matrix} -3 \\ 1 \end{matrix}\right\|$.

IM4) $\left\|\begin{matrix} 3 & -1 & k \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{matrix}\right\| \rightarrow \left|\begin{matrix} 3-\lambda & -1 & k \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{matrix}\right| = \left|\begin{matrix} 2-\lambda & -1 & k \\ 2-\lambda & -5-\lambda & 1 \\ 0 & -6 & 2-\lambda \end{matrix}\right| =$

$C_1 \leftarrow C_1 + C_2$

$= (2-\lambda)\cdot\left|\begin{matrix} -5-\lambda & 1 \\ -6 & 2-\lambda \end{matrix}\right| - (2-\lambda)\cdot\left|\begin{matrix} -1 & k \\ -6 & 2-\lambda \end{matrix}\right| =$

$= (2-\lambda)\cdot\left[(-5-\lambda)(2-\lambda) + 6 - \left(-1(2-\lambda) + 6k\right)\right] =$

$= (2-\lambda)\cdot\left(\lambda^2 + 3\lambda - 10 + 6 + 2 - \lambda - 6k\right) = (2-\lambda)\cdot\left(\lambda^2 + 2\lambda - 2 - 6k\right) = 0$

$\lambda = 2$ eigenvalue $\forall K$. MfEA2

•) for $\lambda = 2$: $4+4-2-6K = 6-6K = 0 \Rightarrow K = 1$.

$\lambda^2 + 2\lambda - 8 = 0 \Rightarrow \lambda = -1 \pm \sqrt{1+8} = -1 \pm 3 = \left< \begin{matrix} -4 \\ 2 \end{matrix} \right.$.

for $K = 1$: $\lambda_1 = \lambda_2 = 2$; $\lambda_3 = -4$.

$$\|A - 2\mathbb{I}\| = \left\|\begin{matrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{matrix}\right\| \rightarrow \left\|\begin{matrix} 0 & -1 & 1 \\ 0 & -7 & 1 \\ 0 & -6 & 0 \end{matrix}\right\| . \left|\begin{matrix} -1 & 1 \\ -7 & 1 \end{matrix}\right| = -1+7 = 6 \neq 0.$$

RANK $(\|A-2\mathbb{I}\|) = 2 \Rightarrow m_2^g = 3-2 = 1 < m_2^a = 2$. A is not diagonalizable.

••) $\lambda^2 + 2\lambda - 2 - 6K = 0 \Rightarrow \lambda = -1 \pm \sqrt{1+2+6K} = -1 \pm \sqrt{3+6K}$. $\Delta = 0$ for $K = -\frac{1}{2}$.

for $K = -\frac{1}{2}$: $\lambda_1 = 2$; $\lambda_2 = \lambda_3 = -1$.

$$\|A - (-1)\mathbb{I}\| = \left\|\begin{matrix} 4 & -1 & -\frac{1}{2} \\ 7 & -4 & 1 \\ 6 & -6 & 3 \end{matrix}\right\| . \left|\begin{matrix} 4 & -1 \\ 7 & -4 \end{matrix}\right| = -16+7 = -9 \neq 0.$$

RANK $(\|A-(-1)\mathbb{I}\|) = 2 \Rightarrow m_{-1}^g = 3-2 = 1 < m_{-1}^a = 2$. A is not diagonalizable.

We have complex eigenvalues if $\Delta < 0 \Rightarrow 3+6K < 0 \Rightarrow K < -\frac{1}{2}$.

IM5) We find a basis for the Kernel of A:

$(1;-1;1)\cdot(x;y;z) = x-y+z = 0 \Rightarrow z = y-x \Rightarrow (x;y;y-x) \in$ Ker.

So a basis is $\{(1;0;-1);(0;1;1)\}$.

$$\left\|\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} x_1 - z_1 = 0 \\ x_2 - z_2 = 0 \\ x_3 - z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = x_1 \\ z_2 = x_2 \\ z_3 = x_3 \end{cases} ; \left\|\begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{matrix}\right\| \cdot \left\|\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} y_1 + z_1 = 0 \\ y_2 + z_2 = 0 \\ y_3 + z_3 = 0 \end{cases} \Rightarrow \begin{cases} y_1 = -z_1 = -x_1 \\ y_2 = -z_2 = -x_2 \\ y_3 = -z_3 = -x_3 \end{cases}$$

$$\left\|\begin{matrix} x_1 & -x_1 & x_1 \\ x_2 & -x_2 & x_2 \\ x_3 & -x_3 & x_3 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ 2 \\ -1 \end{matrix}\right\| = \left\|\begin{matrix} 4 \\ 2 \\ -2 \end{matrix}\right\| \Rightarrow \begin{cases} x_1 - 2x_1 - x_1 = 4 \\ x_2 - 2x_2 - x_2 = 2 \\ x_3 - 2x_3 - x_3 = -2 \end{cases} \Rightarrow \begin{cases} x_1 = -2 \\ x_2 = -1 \\ x_3 = +1 \end{cases} . \quad A = \left\|\begin{matrix} -2 & 2 & -2 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{matrix}\right\|.$$

$$\|A - \lambda\mathbb{I}\| = \left\|\begin{matrix} -2-\lambda & 2 & -2 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{matrix}\right\| \rightarrow \left|\begin{matrix} -2-\lambda & 2 & -2 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{matrix}\right| = \left|\begin{matrix} -2-\lambda & 2 & -2 \\ 0 & -\lambda & -\lambda \\ 1 & -1 & 1-\lambda \end{matrix}\right| =$$

$R_2 \leftarrow R_2 + R_3$

$$= 0 - \lambda \cdot \left|\begin{matrix} -2-\lambda & -2 \\ 1 & 1-\lambda \end{matrix}\right| + \lambda \left|\begin{matrix} -2-\lambda & 2 \\ 1 & -1 \end{matrix}\right| =$$

$$= (-\lambda)(\lambda^2 + \lambda - 2 + 2) + \lambda(2 + \lambda - 2) = (-\lambda)(\lambda^2 + \lambda - \lambda) = -\lambda^3 = 0 \Rightarrow$$

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$$