Task Mathematics for Economic Applications. 27/1/2014 MfEA 1

IM1) $\sqrt[4]{i^{68}-i^{57}} = \sqrt[4]{i^{17\cdot 4}-i^{14\cdot 4+1}} = \sqrt[4]{(i^4)^{17}-(i^4)^{14}\cdot i} = \sqrt[4]{1^{17}-1^{14}\cdot i} = \sqrt[4]{1-i}$.

$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{7}{4}\pi + i \operatorname{sen}\frac{7}{4}\pi\right)$.

$\sqrt[4]{1-i} = \sqrt[8]{2}\left(\cos\left(\frac{7}{16}\pi + k\cdot\frac{2\pi}{4}\right) + i \operatorname{sen}\left(\frac{7}{16}\pi + k\cdot\frac{2\pi}{4}\right)\right); 0 \le k \le 3.$

For $k=0$: $\sqrt[8]{2}\left(\cos\frac{7}{16}\pi + i \operatorname{sen}\frac{7}{16}\pi\right)$; For $k=1$: $\sqrt[8]{2}\left(\cos\frac{15}{16}\pi + i \operatorname{sen}\frac{15}{16}\pi\right)$;

For $k=2$: $\sqrt[8]{2}\left(\cos\frac{23}{16}\pi + i \operatorname{sen}\frac{23}{16}\pi\right)$; For $k=3$: $\sqrt[8]{2}\left(\cos\frac{31}{16}\pi + i \operatorname{sen}\frac{31}{16}\pi\right)$.

IM2) $A = \begin{Vmatrix} 2 & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & 2 \end{Vmatrix} \to \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & k-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & k-\lambda & 1 \\ \lambda-1 & 1 & 2-\lambda \end{vmatrix} =$

$= (1-\lambda)\left((k-\lambda)(2-\lambda)-1\right) + (\lambda-1)(1-k+\lambda) = (1-\lambda)(\lambda^2 - 2\lambda - k\lambda + 2k - 1 - 1 + k - \lambda) =$

$= (1-\lambda)(\lambda^2 - (3+k)\lambda + 3k - 2) = 0.$ $\lambda = 1$ eigenvalue $\forall k$.

•) For $\lambda = 1$: $1-(3+k)+3k-2 = 2k-4 = 0 \Rightarrow k = 2$.

For $k=2$: $\lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4) = 0 \Rightarrow \lambda_1 = \lambda_2 = 1; \lambda_3 = 4$.

••) $\lambda^2 - (3+k)\lambda + 3k - 2 = 0 \Rightarrow \lambda = \frac{(3+k) \pm \sqrt{9+k^2+6k-12k+8}}{2} \Rightarrow$

$\Rightarrow \Delta = k^2 - 6k + 17 = 0 \Rightarrow k = 3 \pm \sqrt{9-17}$: no solutions.

The simmetric matrix $A$ has a multiple eigenvalue $\lambda = 1$ only if $k = 2$. No other values for $k$ bring to multiple eigenvalues.

For $k=2$ and $\lambda=1$: $\|A - 1\cdot \mathbb{I}\| = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix} \Rightarrow x+y+z = 0 \Rightarrow z = -x-y$.

We need two orthogonal eigenvectors. The first, from $(x;y;-x-y)$, if $x=1$ and $y=0 \Rightarrow (1;0;-1)$.

Now we need $(1;0;-1)\cdot(x;y;-x-y) = x+x+y = 0 \Rightarrow y = -2x$ and so:

$(x;y;-x-y) \Rightarrow (x;-2x;x) \Rightarrow$ for $x=1$: $(1;-2;1)$.

For $k=2$ and $\lambda=4$: $\|A - 4\mathbb{I}\| = \begin{Vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{Vmatrix} \to \begin{cases} -2x+y+z=0 \\ x-2y+z=0 \\ x+y-2z=0 \end{cases} \to \begin{cases} z = 2x-y \\ x-2y+2x-y=0 \\ x+y-4x+2y=0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 3x = 3y \Rightarrow y = x \\ z = x \\ x+x-4x+2x=0 \end{cases} \Rightarrow (x;x;x)$. For $x=1$: $(1;1;1)$.

$(1;0;-1) \Rightarrow$ unit vector: $\left(\frac{1}{\sqrt{2}};0;-\frac{1}{\sqrt{2}}\right)$

$(1;-2;1) \Rightarrow$ unit vector: $\left(\frac{1}{\sqrt{6}};-\frac{2}{\sqrt{6}};\frac{1}{\sqrt{6}}\right)$ $\Rightarrow U = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{Vmatrix}$.

$(1;1;1) \Rightarrow$ unit vector: $\left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right)$

IM3) $A \cdot B = \begin{Vmatrix} 2 & 1 & \alpha \\ 1 & 1 & \beta \end{Vmatrix} \cdot \begin{Vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{Vmatrix} = \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \circ \begin{Vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{Vmatrix} + \begin{Vmatrix} \alpha \\ \beta \end{Vmatrix} \cdot \begin{Vmatrix} x_3 & y_3 \end{Vmatrix}$ MfEA2

$\Rightarrow A \cdot B = \mathbb{I}_2 \Rightarrow \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{Vmatrix} + \begin{Vmatrix} \alpha \\ \beta \end{Vmatrix} \cdot \begin{Vmatrix} x_3 & y_3 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \Rightarrow$

$\Rightarrow \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} - \begin{Vmatrix} \alpha x_3 & \alpha y_3 \\ \beta x_3 & \beta y_3 \end{Vmatrix} = \begin{Vmatrix} 1-\alpha x_3 & -\alpha y_3 \\ -\beta x_3 & 1-\beta y_3 \end{Vmatrix}.$

$\begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix}$ has Rank = 2 as $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$.

So we have two linear systems, satisfing Cramer's Theorem:

$\begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \circ \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} = \begin{Vmatrix} 1-\alpha x_3 \\ -\beta x_3 \end{Vmatrix}$ and $\begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} y_1 \\ y_2 \end{Vmatrix} = \begin{Vmatrix} -\alpha y_3 \\ 1-\beta y_3 \end{Vmatrix}.$

The first system has $\infty^1$ solutions ($x_3 \in \mathbb{R}$) and so the second ($y_3 \in \mathbb{R}$)

So the problem has $\infty^2$ solutions.

IM4) $A = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & m \\ 1 & K & 1 \end{Vmatrix} \rightarrow \begin{Vmatrix} 1 & 1 & 1 \\ 0 & 0 & m-1 \\ 0 & K-1 & 0 \end{Vmatrix}.$

For $m=1$ and $K=1$: $RANK(A) = 1 \Rightarrow Dim(Imm) = 1$ and $Dim(Ker) = 3-1 = 2$.

For $m \neq 1$ and $K=1$
For $m=1$ and $K \neq 1$ : $Rank(A) = 2 \Rightarrow Dim(Imm) = 2$ and $Dim(Ker) = 3-2 = 1$.

For $m \neq 1$ and $K \neq 1$ : $Rank(A) = 3 \Rightarrow Dim(Imm) = 3$ and $Dim(Ker) = 3-3 = 0$

For $m=1$ and $K=1$ : $Dim(Ker) = 2$ : Maximum possible dimention.

Basis for Kernel: $\begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \underline{0} \Rightarrow x+y+z=0 \Rightarrow z = -x-y \Rightarrow$

$(x; y; -x-y) \Rightarrow (1; 0; -1)$ and $(0; 1; -1)$ is a basis for the Kernel.

$\begin{Vmatrix} 1 & 1 & 1 & \vdots & y_1 \\ 1 & 1 & 1 & \vdots & y_2 \\ 1 & 1 & 1 & \vdots & y_3 \end{Vmatrix} \rightarrow \begin{Vmatrix} 1 & 1 & 1 & \vdots & y_1 \\ 0 & 0 & 0 & \vdots & y_2-y_1 \\ 0 & 0 & 0 & \vdots & y_3-y_1 \end{Vmatrix} \Rightarrow \begin{cases} y_2 - y_1 = 0 \\ y_3 - y_1 = 0 \end{cases} \Rightarrow \begin{cases} y_2 = y_1 \\ y_3 = y_1 \end{cases} \Rightarrow$

$= (y_1; y_1; y_1) \Rightarrow (1; 1; 1)$ is a basis for the Image.

1) The Jacobian of $F \circ g$ is equal to $\mathcal{J}F(g(t)) \cdot \mathcal{J}g$. $\mathcal{J}F = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}$,

$\mathcal{J}F(g(t)) = \begin{bmatrix} 2\cos t & 2e^t \\ e^t & \cos t \end{bmatrix}$, $\mathcal{J}g = \begin{pmatrix} -\sin t \\ e^t \end{pmatrix}$; from the previuos results follow $\mathcal{J}(F \circ g) = \begin{bmatrix} 2\cos t & 2e^t \\ e^t & \cos t \end{bmatrix} \cdot \begin{pmatrix} -\sin t \\ e^t \end{pmatrix} = \begin{pmatrix} -2\cos t \sin t + 2e^{2t} \\ -\sin t\, e^t + \cos t\, e^t \end{pmatrix}$.

For the equation of tangent line at $t = 0$, we calculate $F(g(0)) = F(1,1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathcal{J}(F \circ g)(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, tangent line at $t = 0$ has parametric equation $r(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

2) The matrix associated to quadratic form $q$ is $A_q = \begin{bmatrix} 1 & -\alpha & 0 \\ -\alpha & 2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$.

I METHOD (by eigenvalues)

Characteristic polinomial of $A_q$ is $|A_q - \lambda I| = \begin{vmatrix} 1-\lambda & -\alpha & 0 \\ -\alpha & 2-\lambda & 0 \\ 0 & 0 & \alpha - \lambda \end{vmatrix} =$

$(\alpha - \lambda)\begin{vmatrix} 1-\lambda & -\alpha \\ -\alpha & 2-\lambda \end{vmatrix} = (\alpha-\lambda)\big[(1-\lambda)(2-\lambda) - \alpha^2\big] =$

$(\alpha - \lambda)(\lambda^2 - 3\lambda + 2 - \alpha^2)$; the three eigenvalues of $A_q$ are $\lambda_1 = \alpha$, $\lambda_2 = \frac{3-\sqrt{1+4\alpha^2}}{2}$ and $\lambda_3 = \frac{3+\sqrt{1+4\alpha^2}}{2}$. Note that $\lambda_3 > 0,\ \forall\alpha$ and so $q$ never will be negative defined or negative semidefined, $q$ is positive defined iff all eingevalues are positive and that is true iff $\alpha > 0 \wedge \sqrt{1+4\alpha^2} < 3$ or $0 < \alpha < \sqrt{2}$, for $\alpha < 0$ or $\alpha > \sqrt{2}$, $\lambda_1$ and $\lambda_2$ have opposite sign, quadratic form $q$ is indefined, when $\alpha = 0$ or $\alpha = \sqrt{2}$ all eingevalues are not negative whit at least one zero $q$ is positive semidefined.

II METHOD (by principal minors)

Matrix $A_q$ has three principal minors of order one: $1$, $2$, $\alpha$; three principal minors of order two: $\begin{vmatrix} 1 & -\alpha \\ -\alpha & 2 \end{vmatrix} = 2 - \alpha^2$, $\begin{vmatrix} 1 & 0 \\ 0 & \alpha \end{vmatrix} = \alpha$, $\begin{vmatrix} 2 & 0 \\ 0 & \alpha \end{vmatrix} = 2\alpha$; and one principal minor of order three: $\begin{vmatrix} 1 & -\alpha & 0 \\ -\alpha & 2 & 0 \\ 0 & 0 & \alpha \end{vmatrix} = \alpha(2-\alpha^2)$. Note that when $\alpha < 0$ or $\alpha > \sqrt{2}$ at least one minor of even order is negative this implies that $q$ is indefined, if $0 < \alpha < \sqrt{2}$ all principal minors are positive and quadratic form is positive defined, at end in case $\alpha = 0$ or $\alpha = \sqrt{2}$ all principal minors are not negative but at least one is zero, $q$ is positive semidefined.

3) Lagrange function of the problem is

$\mathcal{L}(x,y,z,\lambda) = x^2 + y^2 + z^2 - \lambda(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 3)$.

$\nabla\mathcal{L} = \left(2x + \frac{\lambda}{x^2}, 2y + \frac{\lambda}{y^2}, 2z + \frac{\lambda}{z^2}, -(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 3)\right)$. Put the first order condition we have the sistem:

$$\begin{cases} 2x + \frac{\lambda}{x^2} = 0 \\ 2y + \frac{\lambda}{y^2} = 0 \\ 2z + \frac{\lambda}{z^2} = 0 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3 \end{cases} \Rightarrow \begin{cases} 2x^3 + \lambda = 0 \\ 2y^3 + \lambda = 0 \\ 2z^3 + \lambda = 0 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3 \end{cases} \Rightarrow \begin{cases} x = -\sqrt[3]{\lambda/2} \\ y = -\sqrt[3]{\lambda/2} \\ z = -\sqrt[3]{\lambda/2} \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3 \end{cases} \Rightarrow$$

$$\begin{cases} x = -\sqrt[3]{\lambda/2} \\ y = -\sqrt[3]{\lambda/2} \\ z = -\sqrt[3]{\lambda/2} \\ -3\sqrt[3]{2/\lambda} = 3 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \\ z = 1 \\ \lambda = -2 \end{cases}$$

. One point satisfying first order conditions is $P = (1, 1, 1, -2)$.

$$\widetilde{\mathcal{H}} = \begin{bmatrix} 0 & 1/x^2 & 1/y^2 & 1/z^2 \\ 1/x^2 & 2 - \frac{2\lambda}{x^3} & 0 & 0 \\ 1/y^2 & 0 & 2 - \frac{2\lambda}{y^3} & 0 \\ 1/z^2 & 0 & 0 & 2 - \frac{2\lambda}{z^3} \end{bmatrix}$$

, there's only one point satisfying first order conditions can be usefull sostitued point on $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{H}}(P) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 6 & 0 \\ 1 & 0 & 0 & 6 \end{bmatrix}$,

becouse there's only one constraint we must analize third and fourth principal minors of border hessian:

$$\widetilde{\mathcal{H}}_3(P) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 6 & 0 \\ 1 & 0 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ 1 & 0 \end{vmatrix} = -6 - 6 = -12;$$

$$\widetilde{\mathcal{H}}_4(P) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 6 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 0 \\ 1 & 0 & 6 \\ 1 & 0 & 0 \end{vmatrix} + 6\widetilde{\mathcal{H}}_3(P) =$$

$$-\begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} + 6\widetilde{\mathcal{H}}_3(P) = -36 + 6(-12) = -108.$$

Second order conditions:

$\widetilde{\mathcal{H}}_3(P) = -12 < 0$; $\widetilde{\mathcal{H}}_4(P) = -108 < 0$. $P$ minimum.

The proposed problem has minimum equal 3 on point $(1, 1, 1)$.

4) The condition is satisfied on point $P$, $1 \cdot e = 1 \cdot e$; rewritten the condition as $(x+1)e^{x^2+y^2} - y^2\, e^{x+y} = 0$ we calculate the gradient of $f(x,y) = (x+1)e^{x^2+y^2} - y^2\, e^{x+y}$.

$\nabla f = (e^{x^2+y^2} - 2x(x+1)e^{x^2+y^2} - y^2\, e^{x+y}, 2y(x+1)e^{x^2+y^2} - 2ye^{x+y} - y^2\, e^{x+y})$ and $\nabla f(P) = (0, -e)$, by Dini theorem, the condition $f'_y(P) \neq 0$ is sufficient to have a function $y(x)$ defined in implicity form by the proposed condition, with $y'(0) = -\frac{f'_x(P)}{f'_y(P)} = 0$. To find the nature of point $x_0 = 0$ we must calculate $y''(0)$ that again by Dini theorem can be calculated as $y''(0) = -\frac{f''_{xx}(P)\cdot(f'_y(P))^2 - 2f''_{xy}(P)\cdot f'_x(P)\cdot f'_y(P) + f''_{yy}(P)\cdot(f'_x(P))^2}{(f'_y(P))^3} = -\frac{f''_{xx}(P)}{f'_y(P)}$ (note $f'_x(P) = 0$).

$f''_{xx} = 2xe^{x^2+y^2} - 2(x+1)e^{x^2+y^2} - 2xe^{x^2+y^2} - 4x^2(x+1)e^{x^2+y^2} - y^2\, e^{x+y} =$

$-2(x+1)(2x^2+1)e^{x^2+y^2} - y^2\, e^{x+y}$. $f''_{xx}(P) = -2e - e = -3e$.

$y''(0) = -\frac{-3e}{-e} = -3 < 0$. $x_0 = 0$ is a maximum point for $y(x)$.