Task Mathematics for Economic Applications 17/2/14 MfEA1

IM1) $\sqrt[3]{e^{\log 2+i\cdot\frac{\pi}{3}}}=\sqrt[3]{e^{\log 2}\cdot e^{i\frac{\pi}{3}}}=\sqrt[3]{2\left(\cos\frac{\pi}{3}+i\,\mathrm{sen}\,\frac{\pi}{3}\right)}=$

$=\sqrt[3]{2}\cdot\left(\cos\left(\frac{\pi}{9}+K\cdot\frac{2\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{9}+K\cdot\frac{2\pi}{3}\right)\right);0\le K\le 2.$

$K=0: \sqrt[3]{2}\cdot\left(\cos\frac{\pi}{9}+i\cdot\mathrm{sen}\frac{\pi}{9}\right); K=1: \sqrt[3]{2}\left(\cos\frac{7}{9}\pi+i\,\mathrm{sen}\frac{7}{9}\pi\right); K=2: \sqrt[3]{2}\cdot\left(\cos\frac{13}{9}\pi+i\,\mathrm{sen}\frac{13}{9}\pi\right).$

IM2) $A=\left\|\begin{matrix}3&1&1\\2&K&2\\1&4&3\end{matrix}\right\| \Rightarrow \left|\begin{matrix}3-\lambda&1&1\\2&K-\lambda&2\\1&4&3-\lambda\end{matrix}\right|=\left|\begin{matrix}2-\lambda&1&1\\0&K-\lambda&2\\\lambda-2&4&3-\lambda\end{matrix}\right|=$

$=(2-\lambda)\cdot((K-\lambda)(3-\lambda)-8)+(\lambda-2)(2-K+\lambda)=$

$=(2-\lambda)(\lambda^2-3\lambda-K\lambda+3K-8-2+K-\lambda)=(2-\lambda)(\lambda^2-(4+K)\lambda+4K-10)=0.$

$\lambda=2$ is an eigenvalue $\forall K\in\mathbb{R}$.

•) For $\lambda=2$: $4-8-2K+4K-10=2K-14=0\Rightarrow K=7$.

If $K=7$: $\lambda^2-11\lambda+18=(\lambda-2)(\lambda-9)=0\Rightarrow\lambda_1=\lambda_2=2;\ \lambda_3=9.$

$\|A-2\mathbb{I}\|=\left\|\begin{matrix}1&1&1\\2&5&2\\1&4&1\end{matrix}\right\|;\ \left|\begin{matrix}1&1\\2&5\end{matrix}\right|=5-2=3\neq0$: $RANK(A-2\mathbb{I})=2\Rightarrow m_2^g=3-2=1<2=m_2^a$:

the matrix is not a diagonalizable one.

••) $\lambda^2-(4+K)\lambda+4K-10=0\Rightarrow\lambda=\frac{(4+K)\pm\sqrt{16+K^2+8K-16K+40}}{2}=$

$=\frac{(4+K)\pm\sqrt{K^2-8K+56}}{2}\Rightarrow K^2-8K+56=0$: impossible $\forall K\in\mathbb{R}$.

The matrix is not a diagonalizable one only if $K=7$.

IM3) $f:\mathbb{R}^3\to\mathbb{R}^2$; $Y=A\cdot X$; $A=\left\|\begin{matrix}x_1&x_2&x_3\\y_1&y_2&y_3\end{matrix}\right\|$

•) $(1;0;-1)\in\ker$: $\left\|\begin{matrix}x_1&x_2&x_3\\y_1&y_2&y_3\end{matrix}\right\|\cdot\left\|\begin{matrix}1\\0\\-1\end{matrix}\right\|=\left\|\begin{matrix}x_1-x_3\\y_1-y_3\end{matrix}\right\|=\left\|\begin{matrix}0\\0\end{matrix}\right\|\Rightarrow\begin{cases}x_1-x_3=0\\y_1-y_3=0\end{cases}\Rightarrow\begin{cases}x_3=x_1\\y_3=y_1\end{cases}.$

••) $\left\|\begin{matrix}x_1&x_2&x_3\\y_1&y_2&y_3\end{matrix}\right\|\cdot\left\|\begin{matrix}1\\2\\1\end{matrix}\right\|=\left\|\begin{matrix}2\\-4\end{matrix}\right\|\Rightarrow\begin{cases}x_1+2x_2+x_3=2\\y_1+2y_2+y_3=-4\end{cases}\Rightarrow\begin{cases}2x_1+2x_2=2\\2y_1+2y_2=-4\end{cases}\Rightarrow\begin{cases}x_2=1-x_1\\y_2=-2-y_1\end{cases}.$

$A=\left\|\begin{matrix}x_1&1-x_1&x_1\\y_1&-2-y_1&y_1\end{matrix}\right\|\to\left\|\begin{matrix}x_1&1-x_1&0\\y_1&-2-y_1&0\end{matrix}\right\|\to\left\|\begin{matrix}x_1&1&0\\y_1&-2&0\end{matrix}\right\|.$ $C_3\leftarrow C_3-C_1$. $C_2\leftarrow C_2+C_1$.

To have a surjective map we need $RANK(A)=2\Rightarrow$

$\Rightarrow\left|\begin{matrix}x_1&1\\y_1&-2\end{matrix}\right|=-2x_1-y_1\neq0\Rightarrow y_1\neq-2x_1.$

I M4) $A_2 = \left\| \begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix} \right\|$ MfEA2

$$A \cdot \left\| \begin{matrix} 1 \\ 2 \end{matrix} \right\| = (-1) \cdot \left\| \begin{matrix} 1 \\ 2 \end{matrix} \right\| \Rightarrow \left\| \begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix} \right\| \cdot \left\| \begin{matrix} 1 \\ 2 \end{matrix} \right\| = \left\| \begin{matrix} -1 \\ -2 \end{matrix} \right\| \Rightarrow \begin{cases} x_1 + 2y_1 = -1 \\ x_2 + 2y_2 = -2 \end{cases}.$$

$$A \cdot \left\| \begin{matrix} 2 \\ -1 \end{matrix} \right\| = 1 \cdot \left\| \begin{matrix} 2 \\ -1 \end{matrix} \right\| \Rightarrow \left\| \begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix} \right\| \cdot \left\| \begin{matrix} 2 \\ -1 \end{matrix} \right\| = \left\| \begin{matrix} 2 \\ -1 \end{matrix} \right\| \Rightarrow \begin{cases} 2x_1 - y_1 = 2 \\ 2x_2 - y_2 = -1 \end{cases}.$$

$$\begin{cases} x_1 + 2y_1 = -1 \\ 2x_1 - y_1 = 2 \\ x_2 + 2y_2 = -2 \\ 2x_2 - y_2 = -1 \end{cases} \Rightarrow \begin{matrix} x_1 = \dfrac{\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \dfrac{1-4}{-1-4} = \dfrac{3}{5}; \; y_1 = \dfrac{\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \dfrac{2+2}{-1-4} = -\dfrac{4}{5} \\ x_2 = \dfrac{\begin{vmatrix} -2 & 2 \\ -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \dfrac{2+2}{-1-4} = -\dfrac{4}{5}; \; y_2 = \dfrac{\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \dfrac{-1+4}{-1-4} = -\dfrac{3}{5} \end{matrix}.$$

$$A_2 = \left\| \begin{matrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{matrix} \right\|. \quad |A| = \frac{3}{5}\left(-\frac{3}{5}\right) - \left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right) = -\frac{9}{25} - \frac{16}{25} = -1.$$

$$\left\| \begin{matrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{matrix} \right\| \xrightarrow{Adj} \left\| \begin{matrix} -\frac{3}{3} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{matrix} \right\| \xrightarrow{T} \left\| \begin{matrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{matrix} \right\| \xrightarrow{\frac{1}{|A|}} \left\| \begin{matrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{3} \end{matrix} \right\| = A^{-1}.$$

Se $A = A^{-1} \Rightarrow A \cdot A = \mathbb{1}$.

1) The function is continous on $\mathbb{R}^2/(0,0)$ and we must study if it's continous on $(0,0)$. For this goal we consider the limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} + \frac{x^2y^2}{x^2+y^2}\,.$$

Rewritten the limit in polar coordinates we have

$$\lim_{\rho\to 0} \frac{\rho\cos\alpha\rho\sin\alpha}{\rho} + \frac{(\rho\cos\alpha)^2(\rho\sin\alpha)^2}{\rho^2} =$$

$$\lim_{\rho\to 0} \rho\cos\alpha\sin\alpha + \rho^2\cos^2\alpha\sin^2\alpha = 0\,.$$

Now we must verify that the result is uniformly on α, for this goal we consider

$|\rho\cos\alpha\sin\alpha + \rho^2\cos^2\alpha\sin^2\alpha| \leq |\rho\cos\alpha\sin\alpha| + |\rho^2\cos^2\alpha\sin^2\alpha| =$

$\rho|\cos\alpha\sin\alpha| + \rho^2|\cos^2\alpha\sin^2\alpha| \leq \rho + \rho^2 \leq 2\rho, \forall\alpha$ if $\rho \leq 1$, so we can conclude that $\forall\epsilon > 0,\ \rho < \mathit{MAX}(\epsilon/2, 1/2) \Rightarrow |\rho\cos\alpha\sin\alpha + \rho^2\cos^2\alpha\sin^2\alpha| < \epsilon$, and the convergence is uniform and the limit is 0. The function is continuous if $k = 0$. For differentiability we note that the function has continuos partial derivatives on $\mathbb{R}^2/(0,0)$; at point $(0,0)$ we have differentiability iff it exists an affine function $T(x,y)$ such that:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - T(x,y)}{\sqrt{x^2+y^2}} = 0.$$

Rewritten the limit in polar coordinates with $T(x,y) = ax + by$, we have

$$\lim_{\rho\to 0} \frac{\frac{\rho\cos\alpha\rho\sin\alpha}{\rho} + \frac{(\rho\cos\alpha)^2(\rho\sin\alpha)^2}{\rho^2} - (a\rho\cos\alpha + b\rho\sin\alpha)}{\rho} =$$

$$\lim_{\rho\to 0} \big(\cos\alpha\sin\alpha + \rho\cos^2\alpha\sin^2\alpha - (a\cos\alpha + b\sin\alpha)\big) =$$

$$\cos\alpha(\sin\alpha - a) - b\sin\alpha.$$

The limit is not indipendent from α and the function isn't differentiable on $\mathbb{R}^2$.

2) $$\nabla f = \left(\frac{2x}{x^2-y^2}, \frac{-2y}{x^2-y^2}\right),$$

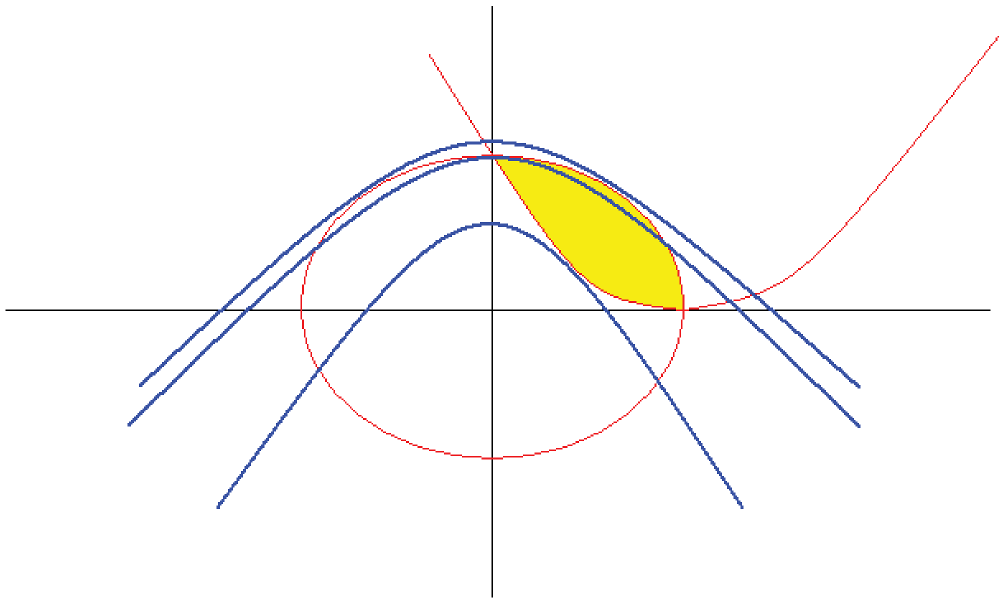
$$\mathcal{H}f = \begin{bmatrix} \frac{2(x^2-y^2)-4x^2}{(x^2-y^2)^2} & \frac{-4xy}{(x^2-y^2)^2} \\ \frac{-4xy}{(x^2-y^2)^2} & \frac{-2(x^2-y^2)-4y^2}{(x^2-y^2)^2} \end{bmatrix} = \begin{bmatrix} -\frac{2(x^2+y^2)}{(x^2-y^2)^2} & \frac{-4xy}{(x^2-y^2)^2} \\ \frac{-4xy}{(x^2-y^2)^2} & -\frac{2(x^2+y^2)}{(x^2-y^2)^2} \end{bmatrix}.$$

$$\|\nabla f\|^2 = \frac{4x^2}{(x^2-y^2)^2} + \frac{4y^2}{(x^2-y^2)^2} = \frac{4(x^2+y^2)}{(x^2-y^2)^2},$$

$$\det(\mathcal{H}f) = \frac{4(x^2+y^2)^2}{(x^2-y^2)^4} - \frac{16x^2y^2}{(x^2-y^2)^4} = \frac{4(x^2-y^2)^2}{(x^2-y^2)^4} = \frac{4}{(x^2-y^2)^2}.$$

Put $\|\nabla f\|^2 = \det(\mathcal{H}f)$ result $\frac{4(x^2+y^2)}{(x^2-y^2)^2} = \frac{4}{(x^2-y^2)^2} \Rightarrow x^2+y^2=1$. Points on $\mathbb{R}^2$ such that $\|\nabla f\|^2 = \det(\mathcal{H}f)$ are points satisfying $x^2+y^2=1$ and $x^2-y^2>0$ that is $-|x| \leq y \leq |x|$.

3) By the grafic below we can observe that admissible region (in yellow, constraints in red) is closed and bounded, objective function (in blue its level curves) is continuous, thus from Weierstrass Theorem the problem admit absolute MAX and min.



Lagrange function of problem is :

$$\Lambda(x,y,\lambda,\mu) = x^2 + y - \lambda(x^2+y^2-1) - \mu((x-1)^2 - y)$$

and its gradient is:

$$\nabla\Lambda = \begin{pmatrix} 2x - 2\lambda x - 2\mu(x-1) \\ 1 - 2\lambda y + \mu \\ -(x^2+y^2-1) \\ -((x-1)^2 - y) \end{pmatrix}.$$

KUHN-TUCKER CONDITIONS

First case: $\begin{cases} \lambda = \mu = 0 \\ 2x = 0 \\ 1 = 0 \\ x^2+y^2-1 \leq 0 \\ (x-1)^2 - y \leq 0 \end{cases}$, the system is impossible.

Second case:

$$\begin{cases} \mu=0 \\ 2x-2\lambda x=0 \\ 1-2\lambda y=0 \\ x^2+y^2-1=0 \\ (x-1)^2-y\le 0 \end{cases} \longrightarrow \begin{cases} \mu=0 \\ 2x(1-\lambda)=0 \\ 1-2\lambda y=0 \\ x^2+y^2=1 \\ (x-1)^2-y\le 0 \end{cases} \begin{matrix} \nearrow \\ \\ \searrow \end{matrix} \begin{matrix} \begin{cases} \mu=0 \\ x=0 \\ \lambda=\pm\frac{1}{2} \\ y=\pm 1 \\ (x-1)^2-y\le 0 \end{cases} \\ \begin{cases} \mu=0 \\ \lambda=1 \\ y=\frac{1}{2} \\ x=\pm\frac{\sqrt{3}}{2} \\ (x-1)^2-y\le 0 \end{cases} \end{matrix}.$$

The up system has only one admissible solution $(0,1,1/2,0)$, as the down system has admissible solution $(\sqrt{3}/2,1/2,1,0)$. Both points are possible MAX.

Third case: $\begin{cases} \lambda=0 \\ 2x-2\mu(x-1)=0 \\ 1+\mu=0 \\ x^2+y^2-1\le 0 \\ (x-1)^2-y=0 \end{cases} \longrightarrow \begin{cases} \lambda=0 \\ x=\frac{1}{2} \\ \mu=-1 \\ x^2+y^2-1\le 0 \\ y=\frac{1}{4} \end{cases}$. One admissible solution is $(1/2,1/4,0,-1)$, a possible min.

Fourth case:

$$\begin{cases} 2x-2\lambda x-2\mu(x-1)=0 \\ 1-2\lambda y+\mu=0 \\ x^2+y^2=1 \\ (x-1)^2=y \end{cases} \longrightarrow \begin{cases} 2x-2\lambda x-2\mu(x-1)=0 \\ 1-2\lambda y+\mu=0 \\ x^2+(x-1)^4=1 \\ (x-1)^2=y \end{cases} \longrightarrow$$

$\begin{cases} 2x-2\lambda x-2\mu(x-1)=0 \\ 1-2\lambda y+\mu=0 \\ x(x-1)(x^2-3x+4)=0 \\ (x-1)^2=y \end{cases}$. The last system presents two solutions, one $(0,1,1/2,0)$ found in case two and one $(1,0,1,-1)$, no MAX, no min because the lagrange multipliers have opposite signs.
$f(0,1)=1$; $f(\sqrt{3}/2,1/2)=5/4$; $f(1/2,1/4)=1/2$; $MAXf=5/4$ on point $(\sqrt{3}/2,1/2)$, $minf=1/2$ on point $(1/2,1/4)$. It's easy to prove by the Jacobian on previous page that the unique active costrain is qualified on both points $(\sqrt{3}/2,1/2)$ and $(1/2,1/4)$.

4) The conditions are satisfied on point P, $\begin{cases} 1^2-(-1)^2+0^2=0 \\ 1(-1)0\,e^{1(-1)0}=0 \end{cases}$; the Jacobian of the functions is

$\mathcal{J}F=\begin{bmatrix} 2x & -2y & 2z \\ yze^{xyz}+xy^2z^2e^{xyz} & xz\,e^{xyz}+x^2yz^2e^{xyz} & xye^{xyz}+x^2y^2z\,e^{xyz} \end{bmatrix}$ and

$\mathcal{J}F(P)=\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Note that Jacobian at point P restricted on couple (x,y) hasn't full rank while restricted on couples (x,z) or (y,z) has; thus this set of conditions can define in a neighbour of point P functions of form $y \mapsto (x(y), z(y))$ or $x \mapsto (y(x), z(x))$. If we choose the first function its derivative at -1 is $\begin{pmatrix} x'(-1) \\ z'(-1) \end{pmatrix} = \begin{pmatrix} -\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \Big/ \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix} \Big/ \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, the parametric form of its tangent line is: $\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; in the second form the derivative at 1 is $\begin{pmatrix} y'(1) \\ z'(1) \end{pmatrix} = \begin{pmatrix} -\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \Big/ \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix} \Big/ \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, the parametric form of its tangent line is: $\begin{pmatrix} x(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.