Task Mathematics for Economic Applications 22/3/14 **MfEA1**

IM1) $2+2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\frac{\pi}{4}+i\,\mathrm{sen}\,\frac{\pi}{4}\right).$

$(2+2i)^{10} = (2\sqrt{2})^{10}\cdot\left(\cos\left(10\cdot\frac{\pi}{4}\right)+i\,\mathrm{sen}\left(10\cdot\frac{\pi}{4}\right)\right) = 2^{15}\cdot\left(\cos\frac{5}{2}\pi+i\,\mathrm{sen}\,\frac{5}{2}\pi\right) = 2^{15}\left(\cos\frac{\pi}{2}+i\,\mathrm{sen}\,\frac{\pi}{2}\right) = 2^{15}\cdot i.$

$\sqrt[3]{(2+2i)^{10}} = 2^5\cdot\left(\cos\left(\frac{\pi}{6}+k\frac{2\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{3}\right)\right);\ 0\le k\le 2.$

if $\underline{k=0}$: $2^5\cdot\left(\cos\frac{\pi}{6}+i\,\mathrm{sen}\,\frac{\pi}{6}\right) = 32\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right);$

if $\underline{k=1}$: $2^5\cdot\left(\cos\frac{5}{6}\pi+i\,\mathrm{sen}\,\frac{5}{6}\pi\right) = 32\cdot\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right);$

if $\underline{k=2}$: $2^5\cdot\left(\cos\frac{9}{6}\pi+i\,\mathrm{sen}\,\frac{9}{6}\pi\right) = 32\cdot\left(\cos\frac{3}{2}\pi+i\,\mathrm{sen}\,\frac{3}{2}\pi\right) = 32\cdot(-i) = -32i.$

IM2) $\left\|\begin{matrix}1&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&1\end{matrix}\right\| \to \begin{vmatrix}1-\lambda&0&0&1\\0&-\lambda&1&0\\0&1&-\lambda&0\\1&0&0&1-\lambda\end{vmatrix} \underset{C_1\leftarrow C_1-C_4}{=} \begin{vmatrix}-\lambda&0&0&1\\0&-\lambda&1&0\\0&1&-\lambda&0\\\lambda&0&0&1-\lambda\end{vmatrix} \underset{R_1\leftarrow R_1+R_4}{=} \begin{vmatrix}0&0&0&2-\lambda\\0&-\lambda&1&0\\0&1&-\lambda&0\\\lambda&0&0&1-\lambda\end{vmatrix} =$

$= -(2-\lambda)\begin{vmatrix}0&-\lambda&1\\0&1&-\lambda\\\lambda&0&0\end{vmatrix} = (\lambda-2)\cdot\lambda\cdot(\lambda^2-1) = \lambda(\lambda-1)(\lambda+1)(\lambda-2) = 0$

$\Rightarrow \lambda_1=0;\ \lambda_2=1;\ \lambda_3=-1;\ \lambda_4=2.$

if $\lambda=0$: $\left\|\begin{matrix}1&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&1\end{matrix}\right\|\cdot\left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases}x_1+x_4=0\\x_3=0\\x_2=0\end{cases} \Rightarrow \begin{cases}x_4=-x_1\\x_3=0\\x_2=0\end{cases}$: Eigenvector: $(x;0;0;-x)$.

if $\lambda=1$: $\left\|\begin{matrix}0&0&0&1\\0&-1&1&0\\0&1&-1&0\\1&0&0&0\end{matrix}\right\|\cdot\left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases}x_4=0\\x_2-x_3=0\\x_1=0\end{cases} \Rightarrow \begin{cases}x_1=0\\x_2=x_3\\x_4=0\end{cases}$: Eigenvector: $(0;x;x;0)$.

if $\lambda=-1$: $\left\|\begin{matrix}2&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&2\end{matrix}\right\|\cdot\left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases}2x_1+x_4=0\\x_2+x_3=0\\x_1+2x_4=0\end{cases} \Rightarrow \begin{cases}x_1=0\\x_3=-x_2\\x_4=0\end{cases}$: Eigenvector: $(0;x;-x;0)$.

if $\lambda=2$: $\left\|\begin{matrix}-1&0&0&1\\0&-2&1&0\\0&1&-2&0\\1&0&0&-1\end{matrix}\right\|\cdot\left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases}x_1-x_4=0\\2x_2-x_3=0\\x_2-2x_3=0\end{cases} \Rightarrow \begin{cases}x_1=x_4\\x_2=0\\x_3=0\end{cases}$: Eigenvector: $(x;0;0;x)$.

To study the quadratic form $X\cdot H\cdot X^T$ it is enough to observe that 2 eigenvalues are positive and 1 eigenvalue is negative. So the quadratic form is not a definite nor semidefinite one.

I.M3) $F(x_1;x_2;x_3;x_4)=(x_1+x_2+x_3+x_4;\ kx_1+x_2+x_3+mx_4;\ x_1+kx_2+mx_3+x_4)$ $\boxed{\text{MfEA2}}$

$$F(x_1;x_2;x_3;x_4)=\begin{Vmatrix}1&1&1&1\\k&1&1&m\\1&k&m&1\end{Vmatrix}\cdot\begin{Vmatrix}x_1\\x_2\\x_3\\x_4\end{Vmatrix}=A\cdot X.$$

If the dimension of the image is equal to 2 $\Rightarrow$ $\mathrm{RANK}(A)=2$.

$$\begin{Vmatrix}1&1&1&1\\k&1&1&m\\1&k&m&1\end{Vmatrix}\xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-k\cdot R_1}\begin{Vmatrix}1&1&1&1\\0&1-k&1-k&m-k\\0&k-1&m-1&0\end{Vmatrix}\xrightarrow{R_3\leftarrow R_3+R_2}\begin{Vmatrix}1&1&1&1\\0&1-k&1-k&m-k\\0&0&m-k&m-k\end{Vmatrix}$$

$\Rightarrow$ $\mathrm{Rank}(A)=2$ iff. $m-k=0\Rightarrow m=k$ and $1-k\neq 0\Rightarrow k\neq 1$.

$G(y_1;y_2;y_3)=(y_1+y_3;\ ky_1+my_2+my_3;\ y_1+y_2+2y_3;\ ky_1+y_2+y_3)$

$$G(y_1;y_2;y_3)=\begin{Vmatrix}1&0&1\\k&m&m\\1&1&2\\k&1&1\end{Vmatrix}\cdot\begin{Vmatrix}y_1\\y_2\\y_3\end{Vmatrix}=B\cdot Y.$$

If the dimension of the kernel is equal to 1 $\Rightarrow$ $3-\mathrm{Rank}(B)=1$ and so we need $\mathrm{Rank}(B)=2$.

$$\begin{Vmatrix}1&0&1\\k&m&m\\1&1&2\\k&1&1\end{Vmatrix}\xrightarrow[\substack{R_3\leftarrow R_3-R_1\\R_4\leftarrow R_4-k\cdot R_1}]{R_2\leftarrow R_2-k\cdot R_1}\begin{Vmatrix}1&0&1\\0&m&m-k\\0&1&1\\0&1&1-k\end{Vmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{Vmatrix}1&0&1\\0&1&1\\0&m&m-k\\0&1&1-k\end{Vmatrix}\xrightarrow[R_4\leftarrow R_4-R_2]{R_3\leftarrow R_3-m\cdot R_2}\begin{Vmatrix}1&0&1\\0&1&1\\0&0&-k\\0&0&-k\end{Vmatrix}.$$

To get $\mathrm{Rank}(B)=2$ we need $k=0$ and so also $m=0$.

$$\text{So } A=\begin{Vmatrix}1&1&1&1\\0&1&1&0\\1&0&0&1\end{Vmatrix}\text{ and } B=\begin{Vmatrix}1&0&1\\0&0&0\\1&1&2\\0&1&1\end{Vmatrix}.$$

To find a basis for the kernel of F: $\mathrm{Dim}(\mathrm{Ker}(F))=4-2=2$.

$$\begin{Vmatrix}1&1&1&1\\0&1&1&0\\1&0&0&1\end{Vmatrix}\cdot\begin{Vmatrix}x_1\\x_2\\x_3\\x_4\end{Vmatrix}=\underline{0}\Rightarrow\begin{cases}x_1+x_2+x_3+x_4=0\\x_2+x_3=0\\x_1+x_4=0\end{cases}\Rightarrow\begin{cases}x_3=-x_2\\x_4=-x_1\end{cases}\Rightarrow$$

$(x_1;x_2;-x_2;-x_4)\Rightarrow\begin{cases}(1;0;0;-1)\text{ and}\\(0;1;-1;0)\text{ is a basis.}\end{cases}$

To find a basis for the Image of G we must satisfy Rouché-Capelli Theorem.

$$\left\Vert\begin{matrix}1&0&1\\0&0&0\\1&1&2\\0&1&1\end{matrix}\ \middle|\ \begin{matrix}z_1\\z_2\\z_3\\z_4\end{matrix}\right\Vert\xrightarrow{R_3\leftarrow R_3-R_1}\left\Vert\begin{matrix}1&0&1\\0&0&0\\0&1&1\\0&1&1\end{matrix}\ \middle|\ \begin{matrix}z_1\\z_2\\z_3-z_1\\z_4\end{matrix}\right\Vert\xrightarrow{R_4\leftarrow R_4-R_3}\left\Vert\begin{matrix}1&0&1\\0&0&0\\0&1&1\\0&0&0\end{matrix}\ \middle|\ \begin{matrix}z_1\\z_2\\z_3-z_1\\z_4-z_3+z_1\end{matrix}\right\Vert.$$

We need so:

$$\begin{cases}z_2=0\\z_4-z_3+z_1=0\end{cases}\Rightarrow\begin{cases}z_2=0\\z_4=z_3-z_1\end{cases}\Rightarrow(z_1;0;z_3;z_3-z_1)<\begin{matrix}(1;0;0;-1)\\(0;0;1;1)\end{matrix}$$

is a basis for $\mathrm{Imm}(G)$.

I.M4) $\begin{Vmatrix}k&k&1-k\\1-k&1-k&1-k\\1-k&k&k\end{Vmatrix}$. To get linearly dependent vectors we need: $\begin{vmatrix}k&k&1-k\\1-k&1-k&1-k\\1-k&k&k\end{vmatrix}=0\Rightarrow$

$$\Rightarrow\begin{vmatrix}k&2k-1&1-k\\1-k&0&1-k\\1-k&0&k\end{vmatrix}_{C_2\leftarrow C_2-C_3}=-(2k-1)\cdot\big(k(1-k)-(1-k)(1-k)\big)=(1-2k)(1-k)(2k-1)=0$$

if $k=\frac{1}{2}$ or $k=1$.

II M1) $f(x;y)=\begin{cases} \dfrac{x^2|y|^\alpha}{(x^2+y^2)^\alpha} & : (x;y)\neq(0;0) \\ 0 & : (x;y)=(0;0) \end{cases}$. $\alpha>0$.

MfEA3

$$\lim_{(x;y)\to(0;0)} \frac{x^2\cdot|y|^\alpha}{(x^2+y^2)^\alpha} \Rightarrow \lim_{\rho\to 0}\frac{\rho^{2+\alpha}\cdot\cos^2\vartheta\cdot|\operatorname{sen}\vartheta|^\alpha}{\rho^{2\alpha}} = 0 \text{ if } 2+\alpha>2\alpha \Rightarrow 0\leq\alpha<2$$

to get a continuous function at $(0;0)$. The convergence is an uniform one as $|\cos^2\vartheta\cdot\operatorname{sen}^\alpha\vartheta|<1$.

$$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to 0}\left(\frac{h^2\cdot 0}{(h^2+0)^\alpha}-0\right)\cdot\frac{1}{h}=\lim_{h\to 0}\frac{0}{h}=0.$$

$$\frac{\partial f}{\partial y}(0;0)=\lim_{h\to 0}\left(\frac{0\cdot|h|^\alpha}{(0+h^2)^\alpha}-0\right)\cdot\frac{1}{h}=\lim_{h\to 0}\frac{0}{h}=0.$$

To have a differentiable function at $(0;0)$ we need:

$$\lim_{(x;y)\to(0;0)}\frac{\frac{x^2|y|^\alpha}{(x^2+y^2)^\alpha}-0-(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}=0 \Rightarrow$$

$$\Rightarrow \lim_{\rho\to 0}\frac{\rho^2\cdot\rho^\alpha\cdot\cos^2\vartheta\cdot|\operatorname{sen}\vartheta|^\alpha}{\rho^{2\alpha}\cdot\rho}=\lim_{\rho\to 0}\frac{\rho^{2+\alpha}\cdot\cos^2\vartheta\cdot|\operatorname{sen}\vartheta|^\alpha}{\rho^{2\alpha+1}}=0$$

iff $2+\alpha>2\alpha+1 \Rightarrow 0\leq\alpha<1$, and so f is a differentiable function.

II M2) $f(x;y)=\frac{1}{2}e^{2-x^2-y^2}$. The function is twice differentiable.

$$\nabla f(x;y)=\left(\tfrac{1}{2}(-2x)\cdot e^{2-x^2-y^2};\ \tfrac{1}{2}(-2y)e^{2-x^2-y^2}\right)=\left(-x\,e^{2-x^2-y^2};\ -y\,e^{2-x^2-y^2}\right).$$

$$\begin{cases} \mathcal{D}_v f(P)=\nabla f(P)\cdot(1;0)=-x\,e^{2-x^2-y^2}=1 \\ \mathcal{D}_w f(P)=\nabla f(P)\cdot(0;1)=-y\,e^{2-x^2-y^2}=-1 \end{cases} \Rightarrow \begin{cases} -x\,e^{2-x^2-y^2}=1 \\ -y\,e^{2-x^2-y^2}=-1 \end{cases} \Rightarrow$$

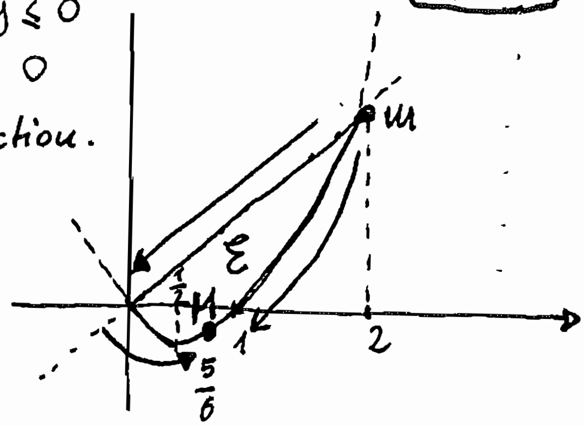
$$\Rightarrow \begin{cases} y=-x \\ -x\,e^{2-2x^2}=1 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=1 \end{cases}$$ is a solution for the problem: $P:(-1;1)$.

$$H(f)=\begin{Vmatrix} -e^{2-x^2-y^2}+2x^2e^{2-x^2-y^2} & 2xy\,e^{2-x^2-y^2} \\ 2xy\,e^{2-x^2-y^2} & -e^{2-x^2-y^2}+2y^2e^{2-x^2-y^2} \end{Vmatrix}=\begin{Vmatrix} (2x^2-1)e^{2-x^2-y^2} & 2xy\,e^{2-x^2-y^2} \\ 2xy\,e^{2-x^2-y^2} & (2y^2-1)e^{2-x^2-y^2} \end{Vmatrix}.$$

$$H(-1;1)=\begin{Vmatrix} 1 & -2 \\ -2 & 1 \end{Vmatrix} \Rightarrow \mathcal{D}^2_{v,w}f(P)=\begin{Vmatrix}1 & 0\end{Vmatrix}\cdot\begin{Vmatrix} 1 & -2 \\ -2 & 1 \end{Vmatrix}\cdot\begin{Vmatrix}0\\1\end{Vmatrix}=\begin{Vmatrix}1;0\end{Vmatrix}\cdot\begin{Vmatrix}-2\\1\end{Vmatrix}=-2.$$

II M3) $\begin{cases} \text{Max/min } f(x;y) = 2x-3y \\ \text{u.e.} \begin{cases} y \geqslant x^2 - x \\ y \leqslant x \end{cases} \end{cases} \Rightarrow \begin{cases} \text{Max/min } f(x;y) = 2x-3y \\ \text{u.e.} \begin{cases} x^2 - x - y \leqslant 0 \\ y - x \leqslant 0 \end{cases} \end{cases}$

MfEA4



$f(x;y)$ is a continuous and differentiable function.

$\mathcal{E}$ is a bounded and closed set.

Constraints are qualified everywhere.

$$\Lambda = 2x - 3y - \lambda_1(x^2 - x - y) - \lambda_2(y - x)$$

For $\lambda_1 = \lambda_2 = 0$

$\begin{cases} \Lambda'_x = 2 \neq 0 \\ \Lambda'_y = -3 \neq 0 \end{cases}$: no solutions

For $\lambda_1 \neq 0;\ \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = 2 - 2\lambda_1 x + \lambda_1 = 0 \\ \Lambda'_y = -3 + \lambda_1 = 0 \\ y = x^2 - x \\ y \leqslant x \end{cases} \Rightarrow \begin{cases} \lambda_1 = 3 \\ 2 - 6x + 3 = 0 \Rightarrow x = \frac{5}{6} \\ y = \frac{25}{36} - \frac{5}{6} = -\frac{5}{36} \\ -\frac{5}{36} \leqslant \frac{5}{6} : \text{satisfied} \end{cases}$$

Max ?

For $\lambda_1 = 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2 + \lambda_2 = 0 \\ \Lambda'_y = -3 - \lambda_2 = 0 \\ y = x \\ y \geqslant x^2 - x \end{cases} \Rightarrow \begin{cases} \lambda_2 = -2 \\ \lambda_2 = -3 \end{cases} : \text{impossible.}$$

For $\lambda_1 \neq 0;\ \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2 - 2\lambda_1 x + \lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = -3 + \lambda_1 - \lambda_2 = 0 \\ \begin{cases} y = x^2 - x \\ y = x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ and } \begin{cases} x = 2 \\ y = 2 \end{cases}. \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 2 + \lambda_1 + \lambda_2 = 0 \\ -3 + \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 2\lambda_1 = 1 \Rightarrow \lambda_1 = \frac{1}{2} > 0 \\ \lambda_2 = -3 + \lambda_1 = -\frac{5}{2} < 0 \end{cases}$$

Nothing

$$\begin{cases} x = 2 \\ y = 2 \\ 2 - 3\lambda_1 + \lambda_2 = 0 \\ -3 + \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 2 \\ 2\lambda_1 = -1 \Rightarrow \lambda_1 = -\frac{1}{2} < 0 \\ \lambda_2 = \lambda_1 - 3 = -\frac{7}{2} < 0 \end{cases}$$

Min ?

From Weierstrass' Theorem $P_1 = \left(\frac{5}{6}; -\frac{5}{36}\right)$ is the absolute **MfEA5**

Maximum point, while $P_2 = (2;2)$ is the absolute minimum point.

II M4) $\begin{cases} f(x;y;z;w) = x^3y - y^2z^2 + zyw^2 = 1 \\ g(x;y;z;w) = e^x y - ze^w + x^2zw = 1 \end{cases}$; $\begin{cases} f(1;1;1;1) = 1 \\ g(1;1;1;1) = 1 \end{cases}$.

$$\frac{\partial(f;g)}{\partial(x;y;z;w)} = \begin{Vmatrix} 3x^2y & x^3 - 2yz^2 + z\cdot w^2 & -2y^2z + yw^2 & 2zyw \\ e^x y + 2xzw & e^x & -e^w + x^2w & -ze^w + x^2z \end{Vmatrix}$$

$$\frac{\partial(f;g)}{\partial(x;y;z;w)}(1;1;1;1) = \begin{Vmatrix} 3 & 0 & -1 & 2 \\ e+2 & e & 1-e & 1-e \end{Vmatrix}$$

Since $\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix} = -1+e-2+2e = 3e-3 \neq 0$, we can define an implicit function $(x;y) \to (z(x;y); w(x;y))$.

$$\frac{\partial z}{\partial x} = -\frac{\begin{vmatrix} 3 & 2 \\ e+2 & 1-e \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{3-3e-2e-4}{3e-3} = \frac{5e+1}{3e-3};$$

$$\frac{\partial z}{\partial y} = -\frac{\begin{vmatrix} 0 & 2 \\ e & 1-e \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{0-2e}{3e-3} = \frac{2e}{3e-3};$$

$$\frac{\partial w}{\partial x} = -\frac{\begin{vmatrix} -1 & 3 \\ 1-e & e+2 \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{-e-2-3+3e}{3e-3} = \frac{5-2e}{3e-3};$$

$$\frac{\partial w}{\partial y} = -\frac{\begin{vmatrix} -1 & 0 \\ 1-e & e \end{vmatrix}}{\begin{vmatrix} -1 & 2 \\ 1-e & 1-e \end{vmatrix}} = -\frac{-e}{3e-3} = \frac{e}{3e-3}.$$