Task Mathematics for economic applications 12/6/2014

I M 1) The reciprocal of $1+\dfrac{1-2i}{2+i}=\dfrac{3-i}{2+i}$ is $\dfrac{2+i}{3-i}=\dfrac{2+i}{3-i}\cdot\dfrac{3+i}{3+i}=\dfrac{5+5i}{10}=\dfrac{1}{2}+\dfrac{1}{2}i$.
Since $\dfrac{1}{2}+\dfrac{1}{2}i=\dfrac{1}{\sqrt{2}}\left(\dfrac{1}{\sqrt{2}}+i\,\dfrac{1}{\sqrt{2}}\right)$, the complex number has argument $\alpha=\dfrac{\pi}{4}$ and modulus $\rho=\sqrt{\left(\dfrac{1}{2}\right)^2+\left(\dfrac{1}{2}\right)^2}=\dfrac{1}{\sqrt{2}}$, so $\dfrac{1}{2}+\dfrac{1}{2}i=\dfrac{1}{\sqrt{2}}\left(\cos\dfrac{\pi}{4}+i\,\mathrm{sen}\,\dfrac{\pi}{4}\right)$.
Computing the square roots we have:

$$\sqrt{\frac{1}{2}+\frac{1}{2}i}=\sqrt{\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4}+i\,\mathrm{sen}\,\frac{\pi}{4}\right)}=\sqrt[4]{\frac{1}{2}}\left(\cos\left(\frac{\frac{\pi}{4}+2k\pi}{2}\right)+i\,\mathrm{sen}\left(\frac{\frac{\pi}{4}+2k\pi}{2}\right)\right)=$$

$$=\sqrt[4]{\frac{1}{2}}\left(\cos\left(\frac{\pi}{8}+k\pi\right)+i\,\mathrm{sen}\left(\frac{\pi}{8}+k\pi\right)\right)\ \text{with } k=0,1.$$

The two roots are $z_0=\sqrt[4]{\dfrac{1}{2}}\left(\cos\dfrac{\pi}{8}+i\,\mathrm{sen}\,\dfrac{\pi}{8}\right)=\sqrt[4]{\dfrac{1}{2}}\left(\dfrac{1}{2}\sqrt{2+\sqrt{2}}+i\,\dfrac{1}{2}\sqrt{2-\sqrt{2}}\right)=$

$=\dfrac{1}{2\sqrt[4]{2}}\left(\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}}\right)$ and

$$z_1=\sqrt[4]{\frac{1}{2}}\left(\cos\frac{9\pi}{8}+i\,\mathrm{sen}\,\frac{9\pi}{8}\right)=\sqrt[4]{\frac{1}{2}}\left(-\frac{1}{2}\sqrt{2+\sqrt{2}}-i\,\frac{1}{2}\sqrt{2-\sqrt{2}}\right)=$$

$$=-\frac{1}{2\sqrt[4]{2}}\left(\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}}\right).$$

Note that $z_1=-z_0$.

I M 2) We put the system in matrix form and using elementary operations on the lines we get:

$$\left\|\begin{matrix}2 & 18 & -8 & | & 20\\ 3 & -3 & 6 & | & -6\\ 4 & 6 & k & | & m\end{matrix}\right\|\xrightarrow[R_2\leftarrow\frac{1}{3}R_2]{R_1\leftarrow\frac{1}{2}R_1}\left\|\begin{matrix}1 & 9 & -4 & | & 10\\ 1 & -1 & 2 & | & -2\\ 4 & 6 & k & | & m\end{matrix}\right\|\xrightarrow[R_3\leftarrow R_3-4R_2]{R_2\leftarrow R_2-R_1}$$

$$\left\|\begin{matrix}1 & 9 & -4 & | & 10\\ 0 & -10 & 6 & | & -12\\ 0 & 10 & k-8 & | & m+8\end{matrix}\right\|\xrightarrow{R_3\leftarrow R_3+R_2}\left\|\begin{matrix}1 & 9 & -4 & | & 10\\ 0 & -10 & 6 & | & -12\\ 0 & 0 & k-2 & | & m-4\end{matrix}\right\|.$$

Using the last matrix we see that:
- if $k\neq 2$, the ranks of $[\mathbb{A}]$ and $[\mathbb{A}|b]$ are both equal to 3; so the system has only one solution;
- if $k=2$ and $m\neq 4$, the rank of $[\mathbb{A}]$ is equal to 2 while the rank of $[\mathbb{A}|b]$ is equal to 3; the system has no solutions;
- if $k=2$ and $m=4$, $[\mathbb{A}]$ and $[\mathbb{A}|b]$ have both rank equal to 2; the system has an infinite number of solutions (∞^1) with one degree of freedom in the choise of the variables.

I M 3) Since $\mathbb{A}$ and $\mathbb{B}$ are similar matrices, they have the same eigenvalues.
From $|\mathbb{A}-\lambda\,\mathbb{I}|=\begin{vmatrix}1-\lambda & 2\\ 4 & -1-\lambda\end{vmatrix}=\lambda^2-9=0$ we get $\lambda_1=3$ and $\lambda_2=-3$.
We have two possible matrices $\mathbb{B}$.
The first is given by solving the system:

$$\begin{cases} \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = 3 \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 3 \\ 3 \end{Vmatrix} \\ \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} = -3 \cdot \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} = \begin{Vmatrix} -3 \\ 0 \end{Vmatrix} \end{cases} \Rightarrow$$

$$\begin{cases} a_{11} + a_{12} = 3 \\ a_{21} + a_{22} = 3 \\ a_{11} + 0 = -3 \\ a_{21} + 0 = 0 \end{cases} \Rightarrow \begin{cases} a_{11} = -3 \\ a_{21} = 0 \\ a_{12} = 6 \\ a_{22} = 3 \end{cases} \Rightarrow \mathbb{B} = \begin{Vmatrix} -3 & 6 \\ 0 & 3 \end{Vmatrix}.$$

The second is given by solving the system:

$$\begin{cases} \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = -3 \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} -3 \\ -3 \end{Vmatrix} \\ \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} = 3 \cdot \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} = \begin{Vmatrix} 3 \\ 0 \end{Vmatrix} \end{cases} \Rightarrow$$

$$\begin{cases} a_{11} + a_{12} = -3 \\ a_{21} + a_{22} = -3 \\ a_{11} + 0 = 3 \\ a_{21} + 0 = 0 \end{cases} \Rightarrow \begin{cases} a_{11} = 3 \\ a_{21} = 0 \\ a_{12} = -6 \\ a_{22} = -3 \end{cases} \Rightarrow \mathbb{B} = \begin{Vmatrix} 3 & -6 \\ 0 & -3 \end{Vmatrix}.$$

I M 4) From $\|\mathbb{A} - \lambda\mathbb{I}\| = \begin{Vmatrix} -7-\lambda & 2 & 5 \\ 2 & -2-\lambda & -2 \\ -5 & 2 & 3-\lambda \end{Vmatrix}$ we compute the determinant:

$$\begin{vmatrix} -7-\lambda & 2 & 5 \\ 2 & -2-\lambda & -2 \\ -5 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 2 & 5 \\ 0 & -2-\lambda & -2 \\ -2-\lambda & 2 & 3-\lambda \end{vmatrix} (C_1 \leftarrow C_1 + C_3)$$

$$= (-2-\lambda)\begin{vmatrix} -2-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} + (-2-\lambda)\begin{vmatrix} 2 & 5 \\ -2-\lambda & -2 \end{vmatrix} =$$

$$= (-2-\lambda)\left[\left(\lambda^2 - \lambda - 2\right) + (6 + 5\lambda)\right] =$$

$$= (-2-\lambda)\left(\lambda^2 + 4\lambda + 4\right) = -(\lambda+2)^3 .$$

The characteristic polynomial of the matrix $\mathbb{A}$ has a multiple solution $\lambda = -2$, whose algebraic multiplicity is equal to 3; since the matrix is not a diagonal one, it is impossible for the matrix to be a diagonalizable one.

In fact, the matrix $\|\mathbb{A} + 2\mathbb{I}\| = \begin{Vmatrix} -5 & 2 & 5 \\ 2 & 0 & -2 \\ -5 & 2 & 5 \end{Vmatrix}$ has rank equal to 2, so the geometric multiplicity of the eingevalue $\lambda = -2$ is equal to 1.

The elements of the eingespace generated by $\lambda = -2$ are the vectors $V = (x, y, z)$ that are solutions of the linear system $\|\mathbb{A} + 2\mathbb{I}\| \cdot V = \mathbb{O}$ or

$$\begin{cases} -5x + 2y + 5z = 0 \\ 2x - 2z = 0 \\ -5x + 2y + 5z = 0 \end{cases} \Rightarrow \begin{cases} -5x + 2y + 5z = 0 \\ 2x - 2z = 0 \end{cases} \Rightarrow \begin{cases} x = z \\ y = 0 \end{cases}.$$

Every eigenvector V has the form $(x, 0, x)$ and a possible base for the eingespace associated to $\lambda = -2$ is $\{(1, 0, 1)\}$, or, using unit vectors, $\left\{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\right\}$.

II M 1) The equation $e^{2x+3y} + 2\,\mathrm{sen}\,(x-y) - 1 = 0$ is satisfied at point $P = (3\pi; -2\pi)$.
If we put $f(x, y) = e^{2x+3y} + 2\,\mathrm{sen}\,(x-y) - 1$ we have:
$\nabla f(x, y) = \left(2e^{2x+3y} + 2\cos(x-y), 3e^{2x+3y} - 2\cos(x-y)\right)$.

At point P we get $\nabla f(P) = (0,5)$ and so we see that in a neighbourhood of P only an implicit function of form $y = y(x)$ can be defined from equation $f(x,y) = 0$.

Its first order derivative is equal to $y'(3\pi) = -\dfrac{f'_x(P)}{f'_y(P)} = 0$.

For calculating its second order derivative we use the formula:

$$y'' = -\frac{f''_{xx}(P) + 2f''_{xy}(P)\cdot y' + f''_{yy}(P)\cdot (y')^2}{f'_y} = -\frac{f''_{xx}(P)}{f'_y} \text{ since } y'(3\pi) = 0.$$

Since $f''_{xx} = 4e^{2x+3y} - 2\,\text{sen}\,(x-y)$, we get $f''_{xx}(P) = 4$ and so $y''(3\pi) = -\dfrac{4}{5}$.

So the requested polynomial is $\mathcal{P}_2(x, 3\pi) = -2\pi - \dfrac{2}{5}(x - 3\pi)^2$.

II M 2) From $f(x,y) = (ax+by)\cdot\cos(x+y)$ we get:

$\nabla f(x,y) = (a\cos(x+y) - (ax+by)\,\text{sen}(x+y); b\cos(x+y) - (ax+by)\,\text{sen}(x+y))$.

So $\nabla f(0,0) = (a,b)$. From condition $\nabla f(0,0) = (1,-2)$ we get $(a,b) = (1,-2)$ and so $f(x,y) = (x-2y)\cos(x+y)$ and finally:

$$g(x,y) = \begin{cases} \dfrac{(x-2y)^2\cos^2(x+y)}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

To check the differentiability of $g(x,y)$ at point $(0,0)$ we consider the limit:

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - g(0,0) - T(x,y)}{\sqrt{x^2+y^2}} \text{ where } T \text{ is a linear operator: } T(x,y) = \alpha x + \beta y.$$

Using polar coordinates, since $g(0,0) = 0$, we have:

$$\lim_{\rho\to 0} \frac{\frac{(\rho\cos\vartheta - 2\rho\,\text{sen}\,\vartheta)^2\cdot\cos^2(\rho\cos\vartheta + \rho\,\text{sen}\,\vartheta)}{\rho} - (\alpha\rho\cos\vartheta + \beta\rho\,\text{sen}\,\vartheta)}{\rho} =$$

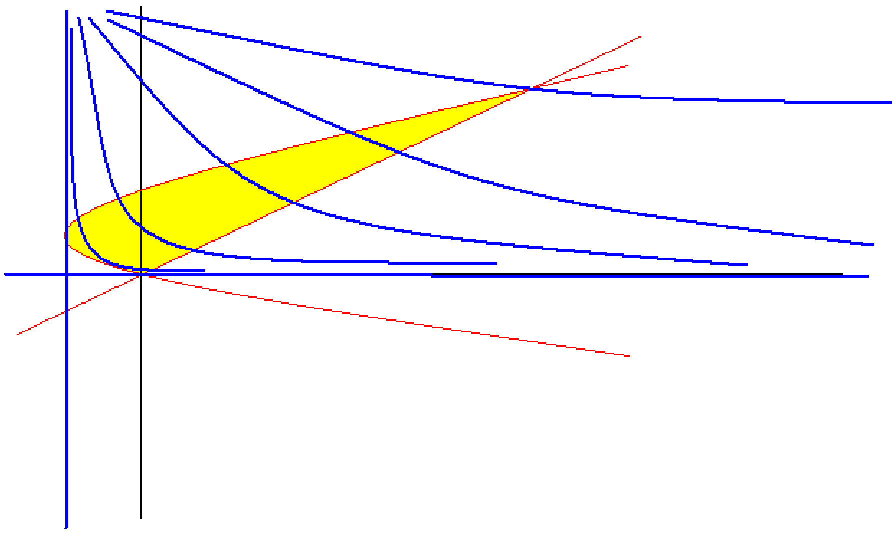
$$\lim_{\rho\to 0} \frac{\frac{\rho^2(\cos\vartheta - 2\text{sen}\,\vartheta)^2\cdot\cos^2(\rho\cos\vartheta + \rho\,\text{sen}\,\vartheta)}{\rho} - \rho(\alpha\cos\vartheta + \beta\,\text{sen}\,\vartheta)}{\rho} =$$

$$\lim_{\rho\to 0} \left[(\cos\vartheta - 2\text{sen}\,\vartheta)^2\cdot\cos^2(\rho\cos\vartheta + \rho\,\text{sen}\,\vartheta) - (\alpha\cos\vartheta + \beta\,\text{sen}\,\vartheta)\right] =$$

$$= (\cos\vartheta - 2\text{sen}\,\vartheta)^2 - (\alpha\cos\vartheta + \beta\,\text{sen}\,\vartheta).$$

The result of the limit is not always equal to zero, so the given function is not differentiable at point $(0,0)$.

II M 3) By the next grafic we can observ that the feasible region $\mathcal{E}$ (in yellow, constraints in red) is closed and bounded, the objective function (in blue its level curves) is continuos, thus from Weierstrass Theorem the problem admits absolute MAX and min. It is easy to prove that in the feasible region $\mathcal{E}$ the Jacobian of the constraints has always full rank and so constraints are everywhere qualified.



The Lagrange function of the problem is :

$$\Lambda(x,y,\lambda,\mu) = xy + y - \lambda\big(y^2 - 2y - x\big) - \mu(x - 2y)$$

and its gradient is:

$$\nabla\Lambda = \begin{pmatrix} y + \lambda - \mu \\ x + 1 - 2\lambda y + 2\lambda + 2\mu \\ -\left(y^2 - 2y - x\right) \\ -(x - 2y) \end{pmatrix}.$$

We solve the problem using Kuhn-Tucker Conditions.

First case: $\lambda = \mu = 0$

$$\begin{cases} \lambda = \mu = 0 \\ y = 0 \\ x + 1 = 0 \\ y^2 - 2y - x \le 0 \\ x - 2y \le 0 \end{cases} \Rightarrow \begin{cases} \lambda = \mu = 0 \\ y = 0 \\ x = -1 \\ 1 \le 0 \\ -1 \le 0 \end{cases} \Rightarrow (-1, 0) \notin \mathcal{E}\,.$$

Second case: $\lambda \neq 0; \mu = 0$

$$\begin{cases} \mu = 0 \\ y + \lambda = 0 \\ x + 1 - 2\lambda y + 2\lambda = 0 \\ y^2 - 2y - x = 0 \\ x - 2y \le 0 \end{cases} \Rightarrow \begin{cases} \mu = 0 \\ y = -\lambda \\ x = -1 - 2\lambda^2 - 2\lambda \\ 3\lambda^2 + 4\lambda + 1 = 0 \\ x - 2y \le 0 \end{cases} \Rightarrow \begin{cases} \mu = 0 \\ y = -\lambda \\ x = -1 - 2\lambda^2 - 2\lambda \\ (3\lambda + 1)(\lambda + 1) = 0 \\ x - 2y \le 0 \end{cases} \Rightarrow$$

$$\begin{cases} \mu = 0 \\ y = \frac{1}{3} \\ x = -\frac{5}{9} \\ \lambda = -\frac{1}{3} \\ -\frac{11}{9} \le 0 \end{cases} \text{ or } \begin{cases} \mu = 0 \\ y = 1 \\ x = -1 \\ \lambda = -1 \\ -3 \le 0 \end{cases}.$$

The first system has the admissible solution $\left(-\frac{5}{9}, \frac{1}{3}, -\frac{1}{3}, 0\right)$, while the second has the admissible solution $(-1, 1, -1, 0)$; both points are possible min.

Third case: $\lambda = 0; \mu \neq 0$

$$\begin{cases} \lambda = 0 \\ y - \mu = 0 \\ x + 1 + 2\mu = 0 \\ x - 2y = 0 \\ y^2 - 2y - x \leq 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = \mu \\ x = -1 - 2\mu \\ -1 - 2\mu - 2\mu = 0 \\ y^2 - 2y - x \leq 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = -\frac{1}{4} \\ x = -\frac{1}{2} \\ \mu = -\frac{1}{4} \\ \frac{17}{16} \leq 0 \end{cases} \Rightarrow \left(-\frac{1}{2}, -\frac{1}{4}\right) \notin \mathcal{E}.$$

Fourt case: $\lambda \neq 0; \mu \neq 0$

$$\begin{cases} y + \lambda - \mu = 0 \\ x + 1 - 2\lambda y + 2\lambda + 2\mu = 0 \\ y^2 - 2y - x = 0 \\ x - 2y = 0 \end{cases} \Rightarrow \begin{cases} y + \lambda - \mu = 0 \\ x + 1 - 2\lambda y + 2\lambda + 2\mu = 0 \\ y^2 - 4y = y(y-4) = 0 \\ x = 2y \end{cases} \Rightarrow$$

$$\begin{cases} \lambda = -\frac{1}{4} \\ \mu = -\frac{1}{4} \\ y = 0 \\ x = 0 \end{cases} \text{or} \begin{cases} \lambda = \frac{17}{4} \\ \mu = \frac{33}{4} \\ y = 4 \\ x = 8 \end{cases}.$$

The first system has the admissible solution $\left(0, 0, -\frac{1}{4}, -\frac{1}{4}\right)$, while the second has the admissible solution $\left(8, 4, \frac{17}{4}, \frac{33}{4}\right)$.

The first point is a possible min, the second is a possible MAX.

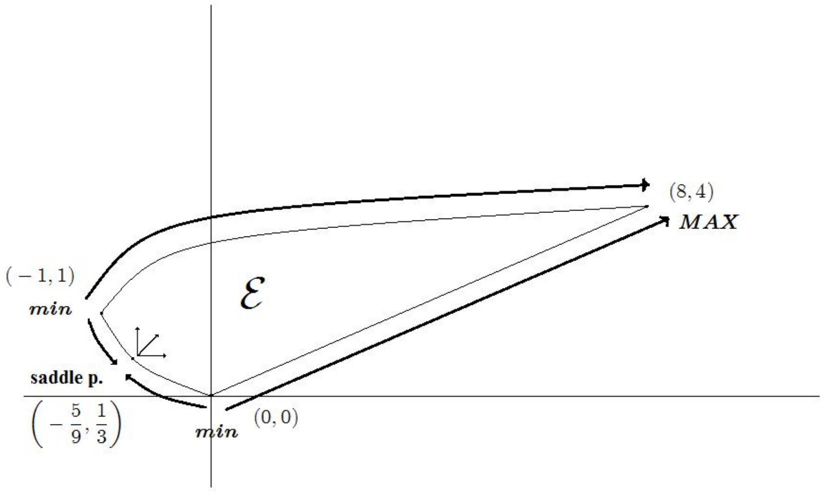
Summarizing, we have found only one possible point of MAX: $(8, 4)$ and three possible points of min: $\left(-\frac{5}{9}, \frac{1}{3}\right)$, $(-1, 1)$, $(0, 0)$.

$(8, 4)$ is the MAX point with $f(8, 4) = 36$.

Examining the objective function on the constraint $x = y^2 - 2y$, by replacing we get:

$f(y) = y^3 - 2y^2 + y \Rightarrow f'(y) = 3y^2 - 4y + 1 \geq 0$ if $0 \leq y \leq \frac{1}{3}$ and if $1 \leq y \leq 4$ while $f'(y) \leq 0$ if $\frac{1}{3} \leq y \leq 1$. With regard to the only border points, $\left(-\frac{5}{9}, \frac{1}{3}\right)$ is a maximum point while by Kuhn-Tucker conditions it may be a minimum point, and so $\left(-\frac{5}{9}, \frac{1}{3}\right)$ is a saddle point.

With regard to the only border points, $(-1, 1)$ and $(0, 0)$ are minimum points as well as by Kuhn-Tucker conditions. Moreover $f(-1, 1) = f(0, 0) = 0$ and so we get two min points.



II M 4) From $g(f(t)) = g(t^2, 1-t) = \left(t^2 - t^3, e^{t^2-t+1}, \dfrac{1}{t^4+t^2-2t+2}\right)$ we get:

$$\frac{d\,g(f(t))}{d\,t} = \left(2t - 3t^2, e^{t^2-t+1}(2t-1), -\frac{4t^3+2t-2}{\left(t^4+t^2-2t+2\right)^2}\right),$$

with $\left.\dfrac{d\,g(f(t))}{d\,t}\right|_{t=0} = \left(0, -e, \dfrac{1}{2}\right)$ and $g(f(0)) = \left(0, e, \dfrac{1}{2}\right)$.

The equation of the tangent line in parametric form is:

$$r(t) = \left(0, e, \frac{1}{2}\right) + t \cdot \left(0, -e, \frac{1}{2}\right) = \left(0, e - et, \frac{1}{2} + \frac{1}{2}t\right).$$