Task Mathematics for Economic Applications 4/7/2014 MfEA1

IM1) $\sqrt{(1-i)^3}$. $1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\sqrt{2}\cdot\left(\cos\frac{7}{4}\pi+i\,\text{sen}\,\frac{7}{4}\pi\right)$.

$(1-i)^3=\sqrt{8}\cdot\left(\cos\frac{21}{4}\pi+i\,\text{sen}\,\frac{21}{4}\pi\right)=2\sqrt{2}\left(\cos\frac{5}{4}\pi+i\,\text{sen}\,\frac{5}{4}\pi\right)$.

$\sqrt{(1-i)^3}=\sqrt[4]{8}\cdot\left(\cos\left(\frac{5}{8}\pi+k\cdot\frac{2\pi}{2}\right)+i\,\text{sen}\left(\frac{5}{8}\pi+k\cdot\frac{2\pi}{2}\right)\right)$; $0\leq k\leq 1$.

For $k=0$: $\sqrt[4]{8}\cdot\left(\cos\frac{5}{8}\pi+i\,\text{sen}\,\frac{5}{8}\pi\right)$;

For $k=1$: $\sqrt[4]{8}\cdot\left(\cos\frac{13}{8}\pi+i\,\text{sen}\,\frac{13}{8}\pi\right)$.

IM2) $A=\left\|\begin{matrix}2&1&1\\0&1&2\\0&0&k\end{matrix}\right\|$ is a triangular matrix, so $\lambda_1=2$; $\lambda_2=1$; $\lambda_3=k$.

If $k\neq1$ and $k\neq2$ we have 3 simple distinct eigenvalues and so the matrix is a diagonalizable one.

If $k=1$: $A=\left\|\begin{matrix}2&1&1\\0&1&2\\0&0&1\end{matrix}\right\|\Rightarrow\left\|A-1\cdot\mathbb{I}\right\|=\left\|\begin{matrix}1&1&1\\0&0&2\\0&0&0\end{matrix}\right\|$: $\text{Rank}\left(A-1\cdot\mathbb{I}\right)=2\Rightarrow m^g_1=3-2=1$.

So $m^g_1=1<m^a_1=2$: the matrix is not a diagonalizable one.

If $k=2$: $A=\left\|\begin{matrix}2&1&1\\0&1&2\\0&0&2\end{matrix}\right\|\Rightarrow\left\|A-2\mathbb{I}\right\|=\left\|\begin{matrix}0&1&1\\0&-1&2\\0&0&2\end{matrix}\right\|$: $\text{Rank}\left(A-2\mathbb{I}\right)=2\Rightarrow m^g_2=3-2=1$.

So $m^g_2=1<m^a_2=2$: the matrix is not a diagonalizable one.

IM3) $A=\left\|\begin{matrix}3&1&1\\2&4&2\\1&m&k\end{matrix}\right\|\Rightarrow\left|A-\lambda\mathbb{I}\right|=\begin{vmatrix}3-\lambda&1&1\\2&4-\lambda&2\\1&m&k-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda&1&1\\0&4-\lambda&2\\1-k+\lambda&m&k-\lambda\end{vmatrix}=$

$=(2-\lambda)\cdot\begin{vmatrix}4-\lambda&2\\m&k-\lambda\end{vmatrix}+(1-k+\lambda)\cdot\begin{vmatrix}1&1\\4-\lambda&2\end{vmatrix}$. If $\lambda=2$: $0\cdot\begin{vmatrix}2&2\\m&k-2\end{vmatrix}+(3-k)\cdot\begin{vmatrix}1&1\\2&2\end{vmatrix}=$

$=0+0=0$. $\lambda=2$ is an eigenvector of A $\forall m$ and $\forall k$.

$\Rightarrow(2-\lambda)\left((4-\lambda)(k-\lambda)-2m\right)+(1-k+\lambda)\cdot(2-4+\lambda)=$

$=(2-\lambda)\left(\lambda^2-k\lambda-4\lambda+4k-2m-1+k-\lambda\right)=(2-\lambda)\cdot\left(\lambda^2-(k+5)\lambda+5k-2m-1\right)=0$

$\Rightarrow\lambda=\dfrac{(k+5)\pm\sqrt{k^2+25+10k-20k+8m+4}}{2}=\dfrac{(k+5)}{2}\pm\dfrac{\sqrt{k^2-10k+8m+29}}{2}$.

If $5\pm i$ is a solution $\Rightarrow\begin{cases}\dfrac{k+5}{2}=5\\ \dfrac{\sqrt{k^2-10k+8m+29}}{2}=i\end{cases}\Rightarrow\begin{cases}k+5=10\Rightarrow k=5\\ k^2-10k+8m+29=-4\end{cases}\Rightarrow$

$\Rightarrow 25-50+8m+33=0\Rightarrow 8m=-8\Rightarrow m=-1$.

The solution is $k=5$ and $m=-1$.

I M4) $\left\|\begin{matrix}1&0&2&-1\\2&1&m&K\\0&K&3&-1\end{matrix}\right\| \xrightarrow{R_2\leftarrow R_2-2R_1} \left\|\begin{matrix}1&0&2&-1\\0&1&m-4&K+2\\0&K&3&-1\end{matrix}\right\| \xrightarrow{R_3\leftarrow R_3-KR_2} \left\|\begin{matrix}1&0&2&-1\\0&1&m-4&K+2\\0&0&3-Km+4K&-(K+1)^2\end{matrix}\right\|.$ $\boxed{\text{MfEA2}}$

$-(K+1)^2=0$ iff $K=-1$.

If $K=-1$: $3+m-4=0$ iff $m=1$.

If $K=-1$ and $m=1$: $\text{Rank}(A)=2 \begin{cases}\text{Dim}(\text{Imm})=2\\ \text{Dim}(\text{Ker})=4-2=2\end{cases}$;

(If $K\neq-1$) OR (If $K=-1$ and $m\neq 1$): $\text{RANK}(A)=3 \begin{cases}\text{Dim}(\text{Imm})=3\\ \text{Dim}(\text{Ker})=4-3=1\end{cases}$.

For $K=-1$ and $m=1$: $A=\left\|\begin{matrix}1&0&2&-1\\2&1&1&-1\\0&-1&3&-1\end{matrix}\right\|$.

To find a basis for the kernel:

$\left\|\begin{matrix}1&0&2&-1\\2&1&1&-1\\0&-1&3&-1\end{matrix}\right\| \to \left\|\begin{matrix}1&0&2&-1\\0&1&-3&1\\0&-1&3&-1\end{matrix}\right\| \to \left\|\begin{matrix}1&0&2&-1\\0&1&-3&1\\0&0&0&0\end{matrix}\right\| \cdot \left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases}x_1+2x_3-x_4=0\\x_2-3x_3+x_4=0\end{cases} \Rightarrow$

$\Rightarrow \begin{cases}x_4=x_1+2x_3\\x_2=3x_3-x_4=3x_3-x_1-2x_3=x_3-x_1\end{cases}$. $V\in\ker: (x_1;x_3-x_1;x_3;x_1+2x_3)$.

For $x_1=1$ and $x_3=0$: $V_1=(1;-1;0;1)$; For $x_1=0$ and $x_3=1$: $V_2=(0;1;1;2)$.

A Basis for the Kernel is $\{(1;-1;0;1);(0;1;1;2)\}$.

To find a basis for the Image:

$\left\|\begin{matrix}1&0&2&-1&\vdots&y_1\\2&1&1&-1&\vdots&y_2\\0&-1&3&-1&\vdots&y_3\end{matrix}\right\| \xrightarrow{R_2\leftarrow R_2-2R_1} \left\|\begin{matrix}1&0&2&-1&\vdots&y_1\\0&1&-3&1&\vdots&y_2-2y_1\\0&-1&3&-1&\vdots&y_3\end{matrix}\right\| \xrightarrow{R_3\leftarrow R_3+R_1} \left\|\begin{matrix}1&0&2&-1&\vdots&y_1\\0&1&-3&1&\vdots&y_2-2y_1\\0&0&0&0&\vdots&y_3+y_2-2y_1\end{matrix}\right\|.$

So we need: $y_3+y_2-2y_1=0 \Rightarrow y_3=2y_1-y_2$. $V\in\text{Imm}=(y_1;y_2;2y_1-y_2)$.

For $y_1=1$ and $y_2=0$: $V_1=(1;0;2)$; For $y_1=0$ and $y_2=1$: $V_2=(0;1;-1)$.

A basis for the Image is $\{(1;0;2);(0;1;-1)\}$.

II M1) $\begin{cases}f(x;y;z)=(x+y+z)\cdot e^{x+y+z}=0\\ g(x;y;z)=x\cdot e^{yz}+y\,e^{xz}+z\cdot e^{xy}=0\end{cases} \Rightarrow \begin{cases}f(1;0;-1)=0\cdot 1=0\\ g(1;0;-1)=1+0-1=0\end{cases}$.

MfEA3

$$\frac{\partial(f;g)}{\partial(x;y;z)}=\left\|\begin{matrix}(x+y+z+1)e^{x+y+z} & (x+y+z+1)e^{x+y+z} & (x+y+z+1)e^{x+y+z}\\ e^{yz}+yze^{xz}+yze^{xy} & xze^{yz}+e^{xz}+xze^{xy} & xye^{yz}+xye^{xz}+e^{xy}\end{matrix}\right\|$$

$$\frac{\partial(f;g)}{\partial(x;y;z)}(1;0;-1)=\left\|\begin{matrix}1 & 1 & 1\\ 1 & \frac{1}{e}-2 & 1\end{matrix}\right\|.$$ Since $\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}=0$ it is not posible to define an implicit function $y\to(x(y);z(y))$.

Since $\begin{vmatrix}1 & 1\\ \frac{1}{e}-2 & 1\end{vmatrix}=1-\frac{1}{e}+2=3-\frac{1}{e}\neq 0$ it is posible to define $F: x\to(y(x);z(x))$.

$$\frac{dy}{dx}=-\frac{\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}}{\begin{vmatrix}1 & 1\\ \frac{1}{e}-2 & 1\end{vmatrix}}=0;\quad \frac{dz}{dx}=-\frac{\begin{vmatrix}1 & 1\\ \frac{1}{e}-2 & 1\end{vmatrix}}{\begin{vmatrix}1 & 1\\ \frac{1}{e}-2 & 1\end{vmatrix}}=-1.$$

The equation of the tangent line at $x=1$: $x\to(0;-1)+x\cdot(0;-1)=(0;-1-x)$.

II M2) $f(x;y)=x\cdot|\operatorname{sen} y|$; $f(0;0)=0$. The function f is continuous at $(0;0)$ since it is given by a product of two continuous functions.

$$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to 0}\frac{(0+h)\cdot|\operatorname{sen} 0|-0}{h}=\lim_{h\to 0}\frac{0}{h}=0;$$

$$\frac{\partial f}{\partial y}(0;0)=\lim_{h\to 0}\frac{0\cdot|\operatorname{sen}(0+h)|-0}{h}=\lim_{h\to 0}\frac{0}{h}=0.$$

For differentiability at $(0;0)$ we must verify if:

$$\lim_{(x;y)\to(0;0)}\frac{x\cdot|\operatorname{sen} y|-0-(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}$$ is or not equal to 0.

Using polar coordinates:

$$\lim_{\rho\to 0}\frac{\rho\cos\vartheta\cdot|\operatorname{sen}(\rho\operatorname{sen}\vartheta)|-0-0}{\rho}=\lim_{\rho\to 0}\cos\vartheta\cdot|\operatorname{sen}(\rho\operatorname{sen}\vartheta)|=0$$

since $\rho\to 0\Rightarrow\rho\cdot\operatorname{sen}\vartheta\to 0\Rightarrow\operatorname{sen}(\rho\operatorname{sen}\vartheta)\to 0$.

Since $|\operatorname{sen}\alpha|\leq|\alpha|\ \forall\alpha\in\mathbb{R}$, it follows that:

$$\big|\cos\vartheta\cdot|\operatorname{sen}(\rho\cdot\operatorname{sen}\vartheta)|-0\big|\leq|\cos\vartheta|\cdot|\operatorname{sen}(\rho\operatorname{sen}\vartheta)|\leq|\cos\vartheta|\cdot|\rho\cdot\operatorname{sen}\vartheta|\leq 1\cdot\rho\cdot 1<\varepsilon.$$

If $\rho<\varepsilon$ the limit is verified, the convergence is uniform and so the function is differentiable at $(0;0)$.

MfEA4

II M3) $\begin{cases} \text{Max } f(x;y;z) = x+y+z \\ \text{u.e.}: x^2+2y^2+3z^2 = 11. \end{cases}$

$$\Lambda = x+y+z - \lambda(x^2+2y^2+3z^2-11)$$

$$\begin{cases} \Lambda'_x = 1-2\lambda x = 0 \\ \Lambda'_y = 1-4\lambda y = 0 \\ \Lambda'_z = 1-6\lambda z = 0 \\ x^2+2y^2+3z^2 = 11 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{4\lambda} \\ z = \frac{1}{6\lambda} \\ \frac{1}{\lambda^2}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{12}\right) = 11 \Rightarrow \lambda^2 = \frac{1}{24} \Rightarrow \lambda = \frac{1}{2\sqrt{6}} \end{cases} \Rightarrow \begin{cases} x = \sqrt{6} \\ y = \frac{\sqrt{6}}{2} \\ z = \frac{\sqrt{6}}{3} \\ \lambda = \frac{1}{2\sqrt{6}} \end{cases}.$$

$$\bar{H}(x;y;z;\lambda) = \begin{Vmatrix} 0 & 2x & 4y & 6z \\ 2x & -2\lambda & 0 & 0 \\ 4y & 0 & -4\lambda & 0 \\ 6z & 0 & 0 & -6\lambda \end{Vmatrix}; \bar{H}\left(\sqrt{6};\frac{\sqrt{6}}{2};\frac{\sqrt{6}}{3};\frac{1}{2\sqrt{6}}\right) = \begin{Vmatrix} 0 & 2\sqrt{6} & 2\sqrt{6} & 2\sqrt{6} \\ 2\sqrt{6} & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 2\sqrt{6} & 0 & -\frac{2}{\sqrt{6}} & 0 \\ 2\sqrt{6} & 0 & 0 & -\frac{3}{\sqrt{6}} \end{Vmatrix}$$

$$|\bar{H}_3(P)| = \begin{vmatrix} 0 & 2\sqrt{6} & 2\sqrt{6} \\ 2\sqrt{6} & -\frac{1}{\sqrt{6}} & 0 \\ 2\sqrt{6} & 0 & -\frac{2}{\sqrt{6}} \end{vmatrix} = \begin{vmatrix} 0 & 2\sqrt{6} & 2\sqrt{6} \\ 0 & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 2\sqrt{6} & 0 & -\frac{2}{\sqrt{6}} \end{vmatrix} = 2\sqrt{6}\cdot(4+2) > 0;$$

$$|\bar{H}_4(P)| = \begin{vmatrix} 0 & 2\sqrt{6} & 2\sqrt{6} & 2\sqrt{6} \\ 2\sqrt{6} & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 2\sqrt{6} & 0 & -\frac{2}{\sqrt{6}} & 0 \\ 2\sqrt{6} & 0 & 0 & -\frac{3}{\sqrt{6}} \end{vmatrix} = \begin{vmatrix} 0 & 2\sqrt{6} & 2\sqrt{6} & 2\sqrt{6} \\ 0 & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 \\ 2\sqrt{6} & 0 & -\frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{3}{\sqrt{6}} \end{vmatrix} = 2\sqrt{6}\cdot 2\sqrt{6}\cdot \begin{vmatrix} \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{6}} \end{vmatrix} = 24\cdot\left(-2+\frac{1}{6}\right) = -44 < 0$$

Since $|\bar{H}_3(P)| > 0$ and $|\bar{H}_4(P)| < 0$, P is a maximum point.

II M4) $f(x;y) = x^3+y^3$: f is a differentiable function $\forall (x;y) \in \mathbb{R}^2$.

$u = \left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right)$; $v = \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$.

$$D_u f(x;y) = \nabla f(x;y)\cdot u = (3x^2;3y^2)\cdot\left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right) = 0;$$
$$D_v f(x;y) = \nabla f(x;y)\cdot v = (3x^2;3y^2)\cdot\left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = 3\sqrt{2}; \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{3}{\sqrt{2}}(x^2-y^2) = 0 \\ \frac{3}{\sqrt{2}}(x^2+y^2) = 3\sqrt{2} \end{cases} \Rightarrow \begin{cases} x^2-y^2 = 0 \\ x^2+y^2 = 2 \end{cases} \Rightarrow \begin{cases} y = \pm x \\ 2x^2 = 2 \end{cases} \Rightarrow \begin{cases} y = \pm x \\ x^2 = 1 \end{cases} \Rightarrow \begin{cases} y = \pm x \\ x = \pm 1 \end{cases}.$$

So we have four solutions: $(1;1)$; $(1;-1)$; $(-1;1)$; $(-1;-1)$.