Task Mathematics for Economic Applications 5/9/2014 $\boxed{\text{MfEA1}}$

IM1) $1+i^{10}-i^{15} = 1+i^{2\cdot4+2}-i^{3\cdot4+3} = 1+(i^4)^2\cdot i^2-(i^4)^3\cdot i^3 =$

$= 1+1\cdot(-1)-1\cdot(-i) = i$; $\quad i = 1\cdot\left(\cos\frac{\pi}{2}+i\,\mathrm{sen}\,\frac{\pi}{2}\right)$.

$\sqrt[3]{i} = \sqrt[3]{1}\cdot\left(\cos\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{3}\right)+i\,\mathrm{sen}\left(\frac{\pi}{6}+k\cdot\frac{2\pi}{3}\right)\right)$; $\quad 0\le k\le 2$.

If $k=0$: $\cos\frac{\pi}{6}+i\,\mathrm{sen}\,\frac{\pi}{6} = \frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}$;

If $k=1$: $\cos\frac{5}{6}\pi+i\,\mathrm{sen}\,\frac{5}{6}\pi = -\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}$;

If $k=2$: $\cos\frac{3}{2}\pi+i\,\mathrm{sen}\,\frac{3}{2}\pi = -i$.

IM2) $A = \begin{Vmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & k \end{Vmatrix} \rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 0 & 1-\lambda & -1 \\ 1 & 2 & k-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1-k+\lambda \\ 0 & 1-\lambda & -1 \\ 1 & 2 & k-\lambda \end{vmatrix} =$

$= (1-\lambda)\big(((1-\lambda)(k-\lambda)+2\big)+1\cdot(\lambda-1)\cdot(1-k+\lambda) = (1-\lambda)\cdot(\lambda^2-k\lambda-\lambda+k+2-1+k-\lambda) =$

$= (1-\lambda)\cdot(\lambda^2-(k+2)\cdot\lambda+2k+1) = 0$. $\quad \lambda=1$ is an eigenvalue $\forall k\in\mathbb{R}$.

•) If $\lambda=1$: $1^2-(k+2)\cdot1+2k+1 = k = 0 \Rightarrow \lambda^2-2\lambda+1 = (\lambda-1)^2 = 0 \Rightarrow$

$\Rightarrow$ For $k=0$: $\lambda_1=\lambda_2=\lambda_3=1$.

••) $\lambda^2-(k+2)\lambda+2k+1=0 \Rightarrow \lambda = \dfrac{k+2\pm\sqrt{(k+2)^2-4(2k+1)}}{2} = \dfrac{k+2\pm\sqrt{k^2-4k}}{2}$.

$\Delta = k^2-4k = k(k-4) = 0$ if $k=0$ and $k=4$.

For $k=0$: $\lambda_1=\lambda_2=\lambda_3=1$.

For $k=4$: $\lambda^2-6\lambda+9 = (\lambda-3)^2 = 0 \Rightarrow \lambda_1=1;\ \lambda_2=\lambda_3=3$.

<u>For $k=0$ and $\lambda=1$</u>: $\|A-1\cdot\mathbb{I}\| = \begin{Vmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 1 & 2 & -1 \end{Vmatrix}$: RANK $=2 \Rightarrow m_1^g = 3-2 = 1 < m_1^a = 3$;

The matrix is not diagonalizable.

For $k=4$ and $\lambda=3$: $\|A-3\mathbb{I}\| = \begin{Vmatrix} -1 & 2 & 1 \\ 0 & -2 & -1 \\ 1 & 2 & 1 \end{Vmatrix}$: RANK $=2 \Rightarrow m_3^g = 3-2 = 1 < m_3^a = 2$;

The matrix is not diagonalizable.

IM3) $\begin{cases} x_1+2x_2-x_4 = 2 \\ 2x_1+x_2-x_3+mx_4 = k \\ -x_1+4x_2+2kx_3+mx_4 = 4 \end{cases} \Rightarrow \left\|\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 2 & 1 & -1 & m & k \\ -1 & 4 & 2k & m & 4 \end{array}\right\| \Rightarrow$

$\Rightarrow \left\|\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & -3 & -1 & m+2 & k-4 \\ 0 & 6 & 2k & m-1 & 6 \end{array}\right\| \Rightarrow \left\|\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & -3 & -1 & m+2 & k-4 \\ 0 & 0 & 2k-2 & 3m+3 & 2k-2 \end{array}\right\|$.

$R_2 \leftarrow R_2-2R_1$ $\qquad$ $R_3 \leftarrow R_3+2R_2$

$R_3 \leftarrow R_3+R_1$

If $k=1$: $\left\|\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & -3 & -1 & m+2 & -3 \\ 0 & 0 & 0 & 3m+3 & 0 \end{array}\right\|$

For $k \neq 1$: $RANK(A) = RANK(A|Y) = 3$: ∞^1 Solutions $\forall m \in \mathbb{R}$.

For $k=1$ and $m=-1$: $RANK(A) = RANK(A|Y) = 2$: ∞^2 Solutions.

For $k=1$ and $m \neq -1$: $RANK(A) = RANK(A|Y) = 3$: ∞^1 Solutions.

I M4) $A = \begin{Vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)\left((1-\lambda)^2-1\right) = (-\lambda)(\lambda^2-2\lambda) = -\lambda^2(\lambda-2) = 0$

$\Rightarrow \lambda_1 = \lambda_2 = 0$; $\lambda_3 = 2$.

For $\lambda = 0$: $\|A - 0\mathbb{I}\| = \|A\| \Rightarrow \begin{Vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x-y=0 \\ \forall z \end{cases} \Rightarrow \begin{cases} y=x \\ \forall z \end{cases} : (x;x;z)$.

We need two eigenvectors : $(1;1;0)$ and $(0;0;1)$.

For $\lambda = 2$: $\|A - 2\mathbb{I}\| = \begin{Vmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x+y=0 \\ 2z=0 \end{cases} \Rightarrow \begin{cases} y=-x \\ z=0 \end{cases} : (x;-x;0)$.

We need one eigenvector : $(1;-1;0)$.

Using unit vectors, the orthogonal matrix is: $U = \begin{Vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{Vmatrix}$.

II M1) $f(x;y;z) = xyz - e^{x-y} + e^{x-z} - e^{y-z} + 1 = 0$. $f(2;2;0) = 0$.

$\nabla f(x;y;z) = \left(yz - e^{x-y} + e^{x-z}\,;\ xz + e^{x-y} - e^{y-z}\,;\ xy - e^{x-z} + e^{y-z}\right)$.

$\nabla f(2;2;0) = \left(0 - 1 + e^2\,;\ 0 + 1 - e^2\,;\ 4 - e^2 + e^2\right) = \left(e^2-1\,;\ 1-e^2\,;\ 4\right)$.

Since $4 \neq 0$ it exists an implicit function : $(x;y) \to z(x;y)$

Equation of the tangent plan: $z - 0 = -\frac{(e^2-1)}{4}(x-2) - \frac{(1-e^2)}{4}(y-2)$.

II M2) $f(x;y) = ax^2 + by^3 + cx^2y^2$. The function is twice differentiable.

$\nabla f(x;y) = (2ax + 2cxy^2\,;\ 3by^2 + 2cx^2y)$. $\nabla f(-1;-1) = (-2a-2c\,;\ 3b-2c)$

$H(f(x;y)) = \begin{Vmatrix} 2a + 2cy^2 & 4cxy \\ 4cxy & 6by + 2cx^2 \end{Vmatrix}$; $H(f(-1;-1)) = \begin{Vmatrix} 2a+2c & 4c \\ 4c & 2c-6b \end{Vmatrix}$.

$f'_x(-1;-1) = -2a - 2c = -4 \Rightarrow a + c = 2 \Rightarrow c = 2 - a$;

$D_v f(-1;-1) = (-2a-2c\,;\ 3b-2c) \cdot \left(\frac{1}{\sqrt{2}}\,;\ \frac{1}{\sqrt{2}}\right) = -3\sqrt{2} \Rightarrow -2a + 3b - 4c = -6 \Rightarrow$

$\Rightarrow 2a-3b+4c=6 \Rightarrow 2a-3b+8-4a=6 \Rightarrow 3b=2-2a \Rightarrow b=\frac{2}{3}(1-a)$; $\boxed{\text{MfEA3}}$

$$D^2_{v,v}f(-1;-1)=\left\|\frac{1}{\sqrt2}\ \frac{1}{\sqrt2}\right\|\cdot\left\|\begin{matrix}2a+2c & 4c\\ 4c & 2c-6b\end{matrix}\right\|\cdot\left\|\begin{matrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{matrix}\right\|=\frac{1}{2}\cdot\|1\ 1\|\cdot\left\|\begin{matrix}2a+2c & 4c\\ 4c & 2c-6b\end{matrix}\right\|\cdot\left\|\begin{matrix}1\\1\end{matrix}\right\|=$$

$$=\frac{1}{2}\cdot\left\|\begin{matrix}2a+6c\\ 6c-6b\end{matrix}\right\|\cdot\left\|\begin{matrix}1\\1\end{matrix}\right\|=7 \Rightarrow 2a+12c-6b=14 \Rightarrow$$

$$\Rightarrow 2a+12(2-a)-6\cdot\frac{2}{3}(1-a)=14 \Rightarrow 2a+24-12a-4+4a=14 \Rightarrow$$

$$\Rightarrow 6a=6 \Rightarrow a=1 \Rightarrow b=0 \Rightarrow c=1.$$

$f(x;y)=x^2+x^2y^2.$

II M3) $\begin{cases}\text{Max/min } f(x;y;z)=x^2+y+z^2\\ \text{u.c.}: x+y^2+z=0\end{cases}$. $\Lambda=x^2+y+z^2-\lambda(x+y^2+z)$

$$\begin{cases}\Lambda'_x=2x-\lambda=0\\ \Lambda'_y=1-2\lambda y=0\\ \Lambda'_z=2z-\lambda=0\\ x+y^2+z=0\end{cases} \Rightarrow \begin{cases}x=\frac{\lambda}{2}\\ y=\frac{1}{2\lambda}\\ z=\frac{\lambda}{2}\\ \frac{\lambda}{2}+\frac{1}{4\lambda^2}+\frac{\lambda}{2}=0\end{cases} \Rightarrow \begin{cases}x=\frac{\lambda}{2}\\ y=\frac{1}{2\lambda}\\ z=\frac{\lambda}{2}\\ \frac{4\lambda^3+1}{4\lambda^2}=0\end{cases} \Rightarrow \begin{cases}x=-\frac{1}{2\sqrt[3]{4}}\\ y=-\frac{\sqrt[3]{4}}{2}\\ z=-\frac{1}{2\sqrt[3]{4}}\\ \lambda=-\frac{1}{\sqrt[3]{4}}\end{cases}.$$

$$\overline{H}=\left\|\begin{matrix}0&1&2y&1\\1&2&0&0\\2y&0&-2\lambda&0\\1&0&0&2\end{matrix}\right\|.\quad |\overline{H}_3|=\begin{vmatrix}0&1&2y\\1&2&0\\2y&0&-2\lambda\end{vmatrix}=(-1)\cdot(-2\lambda)+2y(-4y)=2\lambda-8y^2<0;$$

$$|\overline{H}_4|=\begin{vmatrix}0&1&2y&1\\1&2&0&0\\2y&0&-2\lambda&0\\1&0&0&2\end{vmatrix}=(-1)\begin{vmatrix}1&2y&1\\2&0&0\\0&-2\lambda&0\end{vmatrix}+2\cdot\begin{vmatrix}0&1&2y\\1&2&0\\2y&0&-2\lambda\end{vmatrix}=(-1)(-2)(2\lambda)+2|\overline{H}_3|=4\lambda+2|\overline{H}_3|<0.$$

The point $\left(-\frac{1}{2\sqrt[3]{4}};-\frac{\sqrt[3]{4}}{2};-\frac{1}{2\sqrt[3]{4}}\right)$ is a minimum point.

II M4) Since $\dfrac{\partial(y_1;y_2)}{\partial(t_1;t_2)}(P_0)=\dfrac{\partial(y_1;y_2)}{\partial(x_1;x_2;x_3)}g(P_0)\cdot\dfrac{\partial(x_1;x_2;x_3)}{\partial(t_1;t_2)}(P_0)$ we have:

$$\frac{\partial(y_1;y_2)}{\partial(t_1;t_2)}(P_0)=\left\|\begin{matrix}2&2&-1\\1&0&-1\end{matrix}\right\|\cdot\left\|\begin{matrix}1&2\\2&-1\\-2&0\end{matrix}\right\|=\left\|\begin{matrix}2+4+2 & 4-2+0\\ 1+0+2 & 2+0+0\end{matrix}\right\|=\left\|\begin{matrix}8&2\\3&2\end{matrix}\right\|.$$