Task Mathematics for Economic Applications 23/9/2014 MfEA 1

I M1) $\frac{1-i}{1+i}=\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}=\frac{1-1-2i}{1+1}=-\frac{2i}{2}=-i=1\cdot\left(\cos\frac{3}{2}\pi+i\,\text{sen}\,\frac{3}{2}\pi\right)$.

$\sqrt[4]{\frac{1-i}{1+i}}=\sqrt[4]{-1}=\cos\left(\frac{3}{8}\pi+k\cdot\frac{2\pi}{4}\right)+i\,\text{sen}\left(\frac{3}{8}\pi+k\cdot\frac{2\pi}{4}\right);\ 0\le k\le 3.$

For $k=0$: $\cos\frac{3}{8}\pi+i\,\text{sen}\,\frac{3}{8}\pi$; For $k=1$: $\cos\frac{7}{8}\pi+i\,\text{sen}\,\frac{7}{8}\pi$;

For $k=2$: $\cos\frac{11}{8}\pi+i\,\text{sen}\,\frac{11}{8}\pi$; For $k=3$: $\cos\frac{15}{8}\pi+i\,\text{sen}\,\frac{15}{8}\pi$.

I M2) $|A|=\begin{vmatrix}1&0&1\\1&-1&1\\2&k&0\end{vmatrix}=\begin{vmatrix}0&1&0\\1&-1&1\\2&k&0\end{vmatrix}=(-1)(0-2)=2.$

$|B|=\begin{vmatrix}1&0&0&1\\1&1&0&0\\1&1&1&0\\1&1&1&k\end{vmatrix}=1\begin{vmatrix}1&0&0\\1&1&0\\1&1&k\end{vmatrix}-1\begin{vmatrix}1&1&0\\1&1&1\\1&1&1\end{vmatrix}=1\cdot1\cdot k-0=k.$

The two matrices have the same determinant if $k=2$.

I M3) $X_1=(1;-1;3)$, $X_2=(2;0;2)$, $X_3=(3;k;1)$ $\Rightarrow\begin{vmatrix}1&2&3\\-1&0&k\\3&2&1\end{vmatrix}=\begin{vmatrix}-2&0&2\\-1&0&k\\3&2&1\end{vmatrix}=(-2)\cdot\begin{vmatrix}-2&2\\-1&k\end{vmatrix}=-2\cdot(-2k+2)=0$ if $k=1$.

$R_1\leftarrow R_1-R_3$

The three vectors are linearly dependent vectors if $k=1$. $\Rightarrow (3;1;1)=\alpha(1;-1;3)+\beta(2;0;2)\Rightarrow$
linear combination: $X_3=\alpha X_1+\beta X_2$

$\Rightarrow\begin{cases}\alpha+2\beta=3\\-\alpha+0\beta=1\\3\alpha+2\beta=1\end{cases}\Rightarrow\begin{cases}\alpha=-1\\2\beta=4\\-3+4=1\end{cases}\Rightarrow\beta=2\Rightarrow\alpha=-1;\ \beta=2\Rightarrow X_3=2X_2-X_1.$

I M4) $A=\begin{Vmatrix}1&0&1\\1&1&0\\0&1&1\end{Vmatrix}\to\begin{vmatrix}1-\lambda&0&1\\1&1-\lambda&0\\0&1&1-\lambda\end{vmatrix}=(1-\lambda)\left[(1-\lambda)^2-0\right]+1\cdot(1-0)=(1-\lambda)^3+1=0\Rightarrow$

$\Rightarrow -\lambda^3+3\lambda^2-3\lambda+2=0\Rightarrow\lambda^3-3\lambda^2+3\lambda-2=0$. For $\lambda=2$: $8-12+6-2=0$.

$$\begin{array}{c|ccc|c} & 1 & -3 & 3 & -2\\ 2 & & 2 & -2 & 2\\ \hline & 1 & -1 & 1 & 0\end{array}$$

$\Rightarrow\lambda^3-3\lambda^2+3\lambda-2=(\lambda-2)(\lambda^2-\lambda+1)=0.$

$\lambda^2-\lambda+1=0\Rightarrow\lambda=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{3}\cdot i}{2}$ $\begin{cases}\frac{1}{2}+\frac{\sqrt{3}}{2}i\\\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{cases}$

Having the matrix three distinct eigenvalues, the matrix is a diagonalizable one.

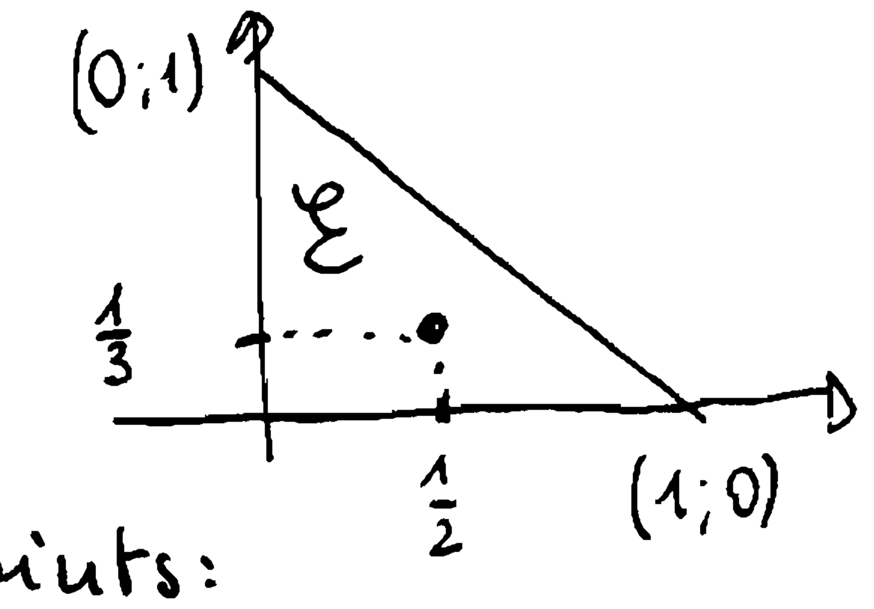
II M1) $f(x;y;z)=e^{x^2y^2z}-e^{xyz^2}=0$; $f(1;1;0)=0$.

$\nabla f(x;y;z)=\left(2xy^2z\,e^{x^2y^2z}-yz^2e^{xyz^2};\,2x^2yz\,e^{x^2y^2z}-xz^2e^{xyz^2};\,x^2y^2e^{x^2y^2z}-2xyz\,e^{xyz^2}\right)$

$\nabla f(1;1;0)=(0\cdot1-0\cdot1;\,0\cdot1-0\cdot1;\,1\cdot1-0\cdot1)=(0;0;1)$. $f'_z(P)=1\neq0$.

$\frac{\partial z}{\partial x}(P) = -\frac{0}{1} = 0;\ \frac{\partial z}{\partial y}(P) = -\frac{0}{1} = 0 \Rightarrow \nabla z(1;1) = (0;0).$

II M2) $\begin{cases} \text{Max/min } f(x;y) = 6xy - 2x - 3y \\ \text{u.c.} \begin{cases} x \geq 0 \\ y \geq 0 \\ y \leq 1 - x \end{cases} \end{cases}$



Free maximum and minimum points:

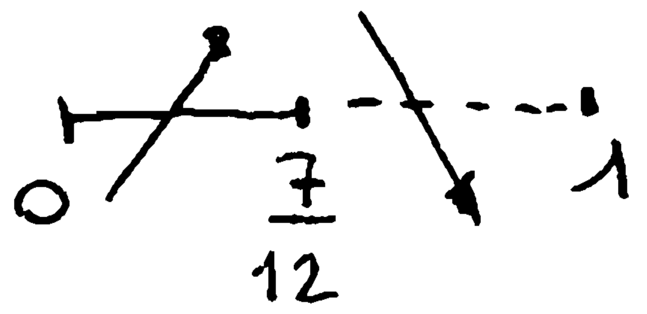
$\begin{cases} f'_x = 6y - 2 = 0 \\ f'_y = 6x - 3 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{1}{3} \end{cases};\ H = \begin{Vmatrix} 0 & 6 \\ 6 & 0 \end{Vmatrix} = H\left(\frac{1}{2};\frac{1}{3}\right): |H| = -36 < 0$: Saddle Point.

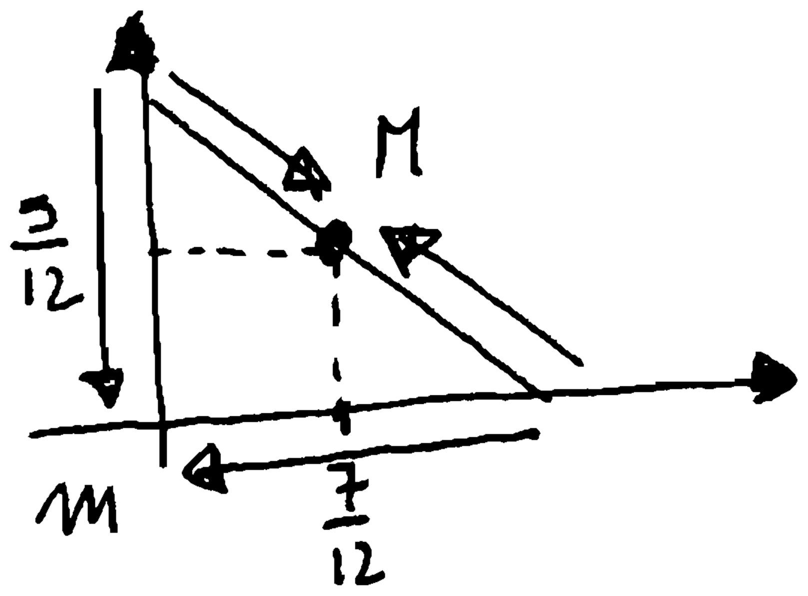
For $y = 0$: $f(x;0) = -2x$: decreasing function;

For $x = 0$: $f(0;y) = -3y$: decreasing function;

For $y = 1-x$: $f(x;1-x) = 6x(1-x) - 2x - 3(1-x) = -6x^2 + 7x - 3$; $f'(x) = -12x + 7 \geq 0 \Rightarrow$

$\Rightarrow x \leq \frac{7}{12}$





Point $(0;0)$ is the minimum point

Point $\left(\frac{7}{12};\frac{5}{12}\right)$ is the maximum point.

II M3) $f(x;y;z) = x^2y - xz^2 - z^2 + x^2$. $\nabla f(x;y;z) = (2xy - z^2 + 2x;\ x^2;\ -2xz - 2z)$.

$H(x;y;z) = \begin{Vmatrix} 2y+2 & 2x & -2z \\ 2x & 0 & 0 \\ -2z & 0 & -2x-2 \end{Vmatrix}$. For $x = 0$: $\begin{Vmatrix} 2y+2 & 0 & -2z \\ 0 & 0 & 0 \\ -2z & 0 & -2 \end{Vmatrix}$. As $-2 \neq 0$ $H \not\equiv \mathbb{O}\ \forall (x;y;z)$.

II M4) $f(x;y) = x^2 + y^2$. $f(x;y)$ is a twice differentiable function.

$\nabla f(x;y) = (2x;2y)$; $H(f(x;y)) = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix}$. $u = (\cos\alpha; \operatorname{sen}\alpha)$; $V = (\cos\beta; \operatorname{sen}\beta)$.

$\mathcal{D}^2_{u,v} f(P) = u \cdot H(P) \cdot v^T = \begin{Vmatrix} \cos\alpha & \operatorname{sen}\alpha \end{Vmatrix} \cdot \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\beta \\ \operatorname{sen}\beta \end{Vmatrix} =$

$= \begin{Vmatrix} 2\cos\alpha & 2\operatorname{sen}\alpha \end{Vmatrix} \cdot \begin{Vmatrix} \cos\beta \\ \operatorname{sen}\beta \end{Vmatrix} = 2\cos\alpha\cos\beta + 2\operatorname{sen}\alpha\operatorname{sen}\beta = 2\cdot\cos(\alpha - \beta).$

So $\mathcal{D}^2_{u,v} f(P) = 0$ if $\cos(\alpha - \beta) = 0 \Rightarrow \alpha - \beta = \frac{\pi}{2}$ or $\alpha - \beta = \frac{3}{2}\pi \Rightarrow$

$\Rightarrow \alpha = \beta + \frac{\pi}{2}$ or $\alpha = \beta + \frac{3}{2}\pi$.