Task Mathematics for Economic Applications 15/11/2014 $\boxed{\text{MfEA1}}$

IM1) $\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{Vmatrix} \to \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^2-0\right) + 1(1) = (1-\lambda)^3+1 = 0.$

$\lambda^3 - 3\lambda^2 + 3\lambda - 2 = 0$; if $\lambda = 2$: $8-12+6-2=0.$

$$\begin{array}{c|ccc|c} & 1 & -3 & 3 & -2 \\ 2 & & 2 & -2 & 2 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

$(\lambda-2)(\lambda^2-\lambda+1) = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot i.$

$2 = 2(\cos 0 + i \operatorname{sen} 0)$; $\frac{1}{2} + \frac{\sqrt{3}}{2} i = 1(\cos \frac{\pi}{3} + i \operatorname{sen} \frac{\pi}{3})$; $\frac{1}{2} - \frac{\sqrt{3}}{2} i = 1 \cdot (\cos \frac{5}{3}\pi + i \operatorname{sen} \frac{5}{3}\pi).$

$2 \cdot \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \cdot \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 2 \cdot 1 \cdot 1 \cdot \left(\cos\left(0 + \frac{\pi}{3} + \frac{5}{3}\pi\right) + i \operatorname{sen}\left(0 + \frac{\pi}{3} + \frac{5}{3}\pi\right)\right) = 2 \cdot (\cos 0 + i \operatorname{sen} 0) = 2.$

IM2) $f(x_1;x_2;x_3) = (x_1 - x_3 ; x_1 + 2x_2 + x_3 ; Kx_3 - x_1).$

$f(X) = \begin{Vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & K \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix}$. $\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & K \end{vmatrix} \to \begin{Vmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & K-1 \end{Vmatrix}$. $\operatorname{Dim}(\operatorname{Ker}(f)) = 1$ iff. $\operatorname{RANK}(A) = 2$ $\Rightarrow K = 1.$

<u>Basis for Kernel</u>: $\begin{Vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \underline{0} \Rightarrow \begin{cases} x_1 - x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = x_1 \\ 2x_2 = -2x_1 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x_2 = -x_1 \\ x_3 = x_1 \end{cases}$. $X = (x ; -x ; x)$. Basis for Kernel: $(1;-1;1)$.

<u>Basis for the Image</u>: $\begin{Vmatrix} 1 & 0 & -1 & : & y_1 \\ 1 & 2 & 1 & : & y_2 \\ -1 & 0 & 1 & : & y_3 \end{Vmatrix} \to \begin{Vmatrix} 1 & 0 & -1 & : & y_1 \\ 0 & 2 & 2 & : & y_2 - y_1 \\ 0 & 0 & 0 & : & y_3 + y_1 \end{Vmatrix}$. $\operatorname{RANK}(A) = \operatorname{RANK}(A|Y)$ iff $y_3 = -y_1$.

$Y = (y_1 ; y_2 ; -y_1)$. Basis for Image $\begin{cases} (1;0;-1) \\ (0;1;0) \end{cases}$.

IM3) $\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 2 \\ -1 \\ 2 \end{Vmatrix} \Rightarrow \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} 2 \\ -1 \\ 2 \end{Vmatrix}$. $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2.$

$\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{Vmatrix} \xrightarrow{\text{Adj}} \begin{Vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{Vmatrix} \xrightarrow{T} \begin{Vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{Vmatrix} \Rightarrow A^{-1} = \begin{Vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}.$

$A^{-1} \cdot \begin{Vmatrix} 2 \\ -1 \\ 2 \end{Vmatrix} = \begin{Vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Vmatrix} \cdot \begin{Vmatrix} 2 \\ -1 \\ 2 \end{Vmatrix} = \begin{Vmatrix} 1 + \frac{1}{2} + 1 \\ 1 - \frac{1}{2} - 1 \\ -1 - \frac{1}{2} + 1 \end{Vmatrix} = \begin{Vmatrix} \frac{5}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{Vmatrix}.$

IM4) $A \cdot P = P \cdot B \Rightarrow B = P^{-1} \cdot A \cdot P.$

$P = \begin{Vmatrix} 1 & 1 \\ 2 & 3 \end{Vmatrix}$; $|P| = 3 - 2 = 1$; $P^{-1} = \begin{Vmatrix} 3 & -1 \\ -2 & 1 \end{Vmatrix}.$

$B = \begin{Vmatrix} 3 & -1 \\ -2 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & -1 \\ -2 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 1 \\ 2 & 3 \end{Vmatrix} = \begin{Vmatrix} 3 & -1 \\ -2 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} -1 & -2 \\ 2 & 4 \end{Vmatrix} = \begin{Vmatrix} -5 & -10 \\ 4 & 8 \end{Vmatrix}.$

II M1) $f(P)=\frac{1}{4}+\frac{1}{4}-\frac{6}{4}+1=0$. Condition is satisfied.

$\nabla f=(2x+6y,\ 2y+6x)$, $\nabla f(P)=(-2,2)$; $y'\left(\frac{1}{2}\right)=-\frac{f'_x(P)}{f'_y(P)}=$

$-\frac{-2}{2}=1$. $Hf=\begin{Vmatrix}2 & 6\\ 6 & 2\end{Vmatrix}$ $y''\left(\frac{1}{2}\right)=-\frac{(f''_{xx}+f''_{xy}\cdot y')\cdot f'_y-f'_x(f''_{yx}+f''_{yy}\cdot y')}{(f'_y)^2}=$

$=-\frac{(2+6\cdot 1)\cdot 2-(-2)(6+2\cdot 1)}{4}=-8$. $P_2(x)=-\frac{1}{2}+y'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)+$

$+\frac{1}{2}y''\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)^2=-\frac{1}{2}+x-\frac{1}{2}-4\left(x^2-x+\frac{1}{4}\right)=-4x^2+5x-2.$

II M2) $\nabla f=(y^2e^x-y,\ 2ye^x-x)$, $Hf=\begin{Vmatrix}y^2e^x & 2ye^x-1\\ 2ye^x-1 & 2e^x\end{Vmatrix}$

FOC: $\nabla f=0$ $\begin{cases}y^2e^x-y=0\\ 2y\cdot e^x-x=0\end{cases}\Rightarrow\begin{cases}\frac{x^2}{4e^{2x}}\cdot \cancel{e^x}-\frac{x}{2e^x}=0\\ y=x/(2e^x)\end{cases}\Rightarrow\begin{cases}\frac{x^2-2x}{4e^x}=0\\ y=x/(2e^x)\end{cases}$

$\Rightarrow\begin{cases}x(x-2)=0\\ y=x/(2e^x)\end{cases}\Rightarrow\begin{cases}x=0\\ y=0\end{cases}\vee\begin{cases}x=2\\ y=1/e^2\end{cases}.$

SOC: $Hf(0,0)=\begin{Vmatrix}0 & -1\\ -1 & 2\end{Vmatrix}$, $|Hf(0,0)|=-1<0$ $(0,0)$ saddle.

$Hf(2,1/e^2)=\begin{Vmatrix}1/e^2 & 2/e^2-1\\ 2/e^2-1 & 2e^2\end{Vmatrix}$, $|Hf(2,1/e^2)|=2-\left(\frac{2}{e^2}-1\right)^2>0$

$f''_{xx}(2,1/e^2)=\frac{1}{e^2}>0$, $(2,1/e^2)$ min.

II M3) $\Lambda(x,y,z,\lambda)=x^2-y+z^2-\lambda(x-y^2+z)$

$\nabla\Lambda=(2x-\lambda,\ -1+2\lambda y,\ 2z-\lambda,\ -(x-y^2+z))$,

$\tilde{H}=\begin{Vmatrix}0 & 1 & -2y & 1\\ 1 & 2 & 0 & 0\\ -2y & 0 & 2\lambda & 0\\ 1 & 0 & 0 & 2\end{Vmatrix}$. FOC: $\nabla\Lambda=0$

$\begin{cases}2x-\lambda=0\\ -1+2\lambda y=0\\ 2z-\lambda=0\\ x-y^2+z=0\end{cases}\Rightarrow$

$$\begin{cases} x = \lambda/2 \\ y = 1/(2\lambda) \\ z = x \\ \dfrac{\lambda}{2} - \dfrac{1}{4\lambda^2} + \dfrac{\lambda}{2} = 0 \end{cases} \Rightarrow \begin{cases} x = \lambda/2 \\ y = 1/(2\lambda) \\ z = x \\ \dfrac{4\lambda^3 - 1}{4\lambda^2} = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{4}\sqrt[3]{2} \\ y = \frac{1}{2}\sqrt[3]{4} \\ z = \frac{1}{4}\sqrt[3]{2} \\ \lambda = \frac{1}{2}\sqrt[3]{2} \end{cases} . \; P = \begin{pmatrix} \sqrt[3]{2}/4 \\ \sqrt[3]{4}/2 \\ \sqrt[3]{2}/4 \\ \sqrt[3]{2}/2 \end{pmatrix}$$

SOC: $\tilde{H}(P) = \left\| \begin{matrix} 0 & 1 & -\sqrt[3]{4} & 1 \\ 1 & 2 & 0 & 0 \\ -\sqrt[3]{4} & 0 & \sqrt[3]{2} & 0 \\ 1 & 0 & 0 & 2 \end{matrix} \right\|$;

$$|\tilde{H}_3(P)| = \begin{vmatrix} 0 & 1 & -\sqrt[3]{4} \\ 1 & 2 & 0 \\ -\sqrt[3]{4} & 0 & \sqrt[3]{2} \end{vmatrix} = -\begin{vmatrix} 1 & -\sqrt[3]{4} \\ 0 & \sqrt[3]{2} \end{vmatrix} + 2\begin{vmatrix} 0 & -\sqrt[3]{4} \\ -\sqrt[3]{4} & \sqrt[3]{2} \end{vmatrix} =$$

$$-\sqrt[3]{2} - 4\sqrt[3]{2} = -5\sqrt[3]{2} < 0.$$

$$|\tilde{H}_4(P)| = \begin{vmatrix} 0 & 1 & -\sqrt[3]{4} & 1 \\ 1 & 2 & 0 & 0 \\ -\sqrt[3]{4} & 0 & \sqrt[3]{2} & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -\sqrt[3]{4} & 1 \\ 2 & 0 & 0 \\ 0 & \sqrt[3]{2} & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 & -\sqrt[3]{4} \\ 1 & 2 & 0 \\ -\sqrt[3]{4} & 0 & \sqrt[3]{2} \end{vmatrix} =$$

$$2\begin{vmatrix} -\sqrt[3]{4} & 1 \\ \sqrt[3]{2} & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & -\sqrt[3]{4} \\ 0 & \sqrt[3]{2} \end{vmatrix} + 4\begin{vmatrix} 0 & -\sqrt[3]{4} \\ -\sqrt[3]{4} & \sqrt[3]{2} \end{vmatrix} = -2\sqrt[3]{2} - 2\sqrt[3]{2} - 8\sqrt[3]{2} =$$

$= -12\sqrt[3]{2} < 0$. P min.

$$\min f = f\left(\frac{\sqrt[3]{2}}{4}, \frac{\sqrt[3]{4}}{2}, \frac{\sqrt[3]{2}}{4}\right) = \frac{\sqrt[3]{4} - 4\sqrt[3]{2}}{8}$$ on point $\left(\frac{\sqrt[3]{2}}{4}, \frac{\sqrt[3]{4}}{2}, \frac{\sqrt[3]{2}}{4}\right)$.

II M4) I° METHOD (By eingevalues)

$$P_H(\lambda) = |H - \lambda I| = \begin{vmatrix} K-\lambda & 0 & 1 \\ 0 & K-\lambda & 0 \\ 1 & 0 & K-\lambda \end{vmatrix} = (K-\lambda)\begin{vmatrix} K-\lambda & 1 \\ 1 & K-\lambda \end{vmatrix} =$$

$(K-\lambda)\left[(K-\lambda)^2-1\right]=(K-\lambda)(K-\lambda-1)(K-\lambda+1)$. $\lambda_1=K$, $\lambda_2=K-1$, $\lambda_3=K+1$. In the following diagramm are explained the signs of eingenvalues as a function of K:

K		-1		0		1	
λ_1	$-$	$-$	$-$	0	$+$	$+$	$+$
λ_2	$-$	$-$	$-$	$-$	$-$	0	$+$
λ_3	$-$	0	$+$	$+$	$+$	$+$	$+$

A quadratic form is positive defined iff all the eingenvalues are positive and this is true iff $K>1$; negative defined iff all the eingenvalues are negative and this is true iff $K<-1$.

$\underline{II}^0$ METHOD (By leading minors)

All the leading minors of order one $A_1^1=A_2^1=A_3^1=K$.
The first and the thirth leading minors of order two $A^2_{1,2}=A^2_{2,3}=\begin{vmatrix} K & 0 \\ 0 & K \end{vmatrix}=K^2$; the second $A^2_{1,3}=\begin{vmatrix} K & 1 \\ 1 & K \end{vmatrix}=K^2-1$.
The leading minor of order three $A^3=\begin{vmatrix} K & 0 & 1 \\ 0 & K & 0 \\ 1 & 0 & K \end{vmatrix}=K(K^2-1)$.
Following diagram explains the signs of all the leading minors as a funcion of K:

K		-1		0		1	
A^1_j	$-$	$-$	$-$	0	$+$	$+$	$+$
$A^2_{1,2}$ $A^2_{2,3}$	$+$	$+$	$+$	0	$+$	$+$	$+$
$A^2_{1,3}$	$+$	0	$-$	$-$	$-$	0	$+$
A^3	$-$	0	$+$	0	$-$	0	$+$

a quadratic form is positive defined iff all its leading minors are positive and this is true iff $K > 1$; negative defined iff all odd leading minors are negative and all even principal minors are positive and this is true iff $K < -1$.