TASK MATHEMATICS for ECONOMIC APPLICATIONS 19/1/2015 $\boxed{\text{MfEA1}}$

IM1) $a+b=2;\ a\cdot b=4 \Rightarrow x^2-2x+4=0 \Rightarrow x=1\pm\sqrt{1-4}=1\pm\sqrt{3}\cdot i$

•) $1+\sqrt{3}\cdot i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\cdot\left(\cos\frac{\pi}{3}+i\,\text{sen}\,\frac{\pi}{3}\right).$

$\sqrt{1+\sqrt{3}\cdot i}=\sqrt{2}\left(\cos\left(\frac{\pi}{6}+K\cdot\frac{2\pi}{2}\right)+i\,\text{sen}\left(\frac{\pi}{6}+K\cdot\frac{2\pi}{2}\right)\right);\ 0\le K\le 1.$

$K=0:\ \sqrt{2}\left(\cos\frac{\pi}{6}+i\,\text{sen}\,\frac{\pi}{6}\right)=\sqrt{2}\cdot\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right);\ K=1:\ \sqrt{2}\cdot\left(\cos\frac{7}{6}\pi+i\,\text{sen}\,\frac{7}{6}\pi\right)=\sqrt{2}\cdot\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right).$

••) $1-\sqrt{3}\cdot i=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2\left(\cos\frac{5}{3}\pi+i\,\text{sen}\,\frac{5}{3}\pi\right).$

$\sqrt{1-\sqrt{3}i}=\sqrt{2}\left(\cos\left(\frac{5}{6}\pi+K\cdot\frac{2\pi}{2}\right)+i\,\text{sen}\left(\frac{5}{6}\pi+K\cdot\frac{2\pi}{2}\right)\right);\ 0\le K\le 1.$

$K=0:\ \sqrt{2}\cdot\left(\cos\frac{5}{6}\pi+i\,\text{sen}\,\frac{5}{6}\pi\right)=\sqrt{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right);\ K=1:\ \sqrt{2}\left(\cos\frac{11}{6}\pi+i\,\text{sen}\,\frac{11}{6}\pi\right)=\sqrt{2}\cdot\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right).$

IM2) $A=\begin{Vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{Vmatrix};\ \begin{Vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{Vmatrix}\cdot\begin{Vmatrix}1\\1\\1\end{Vmatrix}=\begin{Vmatrix}0\\0\\0\end{Vmatrix}\Rightarrow\begin{cases}x_1+y_1+z_1=0\\x_2+y_2+z_2=0\\x_3+y_3+z_3=0\end{cases}\Rightarrow\begin{cases}z_1=-x_1-y_1\\z_2=-x_2-y_2\\z_3=-x_3-y_3\end{cases};$

$\begin{Vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{Vmatrix}\cdot\begin{Vmatrix}1\\1\\0\end{Vmatrix}=\begin{Vmatrix}0\\1\\-1\end{Vmatrix}\Rightarrow\begin{Vmatrix} x_1 & y_1 & -x_1-y_1\\ x_2 & y_2 & -x_2-y_2\\ x_3 & y_3 & -x_3-y_3\end{Vmatrix}\cdot\begin{Vmatrix}1\\1\\0\end{Vmatrix}=\begin{Vmatrix}0\\1\\-1\end{Vmatrix}\Rightarrow\begin{cases}x_1+y_1=0\\x_2+y_2=1\\x_3+y_3=-1\end{cases}\Rightarrow\begin{cases}y_1=-x_1\\y_2=1-x_2\\y_3=-1-x_3\end{cases};$

$\begin{Vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{Vmatrix}\cdot\begin{Vmatrix}1\\-1\\1\end{Vmatrix}=\begin{Vmatrix}2\\-4\\6\end{Vmatrix}\Rightarrow\begin{Vmatrix} x_1 & -x_1 & 0\\ x_2 & 1-x_2 & -1\\ x_3 & -1-x_3 & 1\end{Vmatrix}\cdot\begin{Vmatrix}1\\-1\\1\end{Vmatrix}=\begin{Vmatrix}2\\-4\\6\end{Vmatrix}\Rightarrow\begin{cases}2x_1=2\\2x_2-2=-4\\2x_3+2=6\end{cases}\Rightarrow\begin{cases}x_1=1\\x_2=-1\\x_3=2\end{cases}\Rightarrow$

$\Rightarrow A=\begin{Vmatrix}1 & -1 & 0\\-1 & 2 & -1\\2 & -3 & 1\end{Vmatrix}.\ |A-\lambda\mathbb{I}|=\begin{vmatrix}1-\lambda & -1 & 0\\-1 & 2-\lambda & -1\\2 & -3 & 1-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda & -1 & 0\\0 & 2-\lambda & -1\\1+\lambda & -3 & 1-\lambda\end{vmatrix}=(1-\lambda)(\lambda^2-3\lambda-1)+(1+\lambda)=$

$=\lambda^2-3\lambda-1-\lambda^3+3\lambda^2+\lambda+1+\lambda=-\lambda^3+4\lambda^2-\lambda=(-\lambda)\cdot(\lambda^2-4\lambda+1)=0.\ \lambda_1=0;$

$\lambda_2;\lambda_3=2\pm\sqrt{4-1}=2\pm\sqrt{3}$. Eigenvalues of A: $\lambda_1=0;\ \lambda_2=2+\sqrt{3};\ \lambda_3=2-\sqrt{3}.$

IM3) $A=\begin{Vmatrix}1&0&1\\0&K&0\\1&0&1\end{Vmatrix}\rightarrow|A-\lambda\mathbb{I}|=\begin{vmatrix}1-\lambda&0&1\\0&K-\lambda&0\\1&0&1-\lambda\end{vmatrix}=(K-\lambda)\left((1-\lambda)^2-1\right)=(K-\lambda)\cdot\lambda\cdot(\lambda-2)=0.$

$\lambda_1=K;\ \lambda_2=0;\ \lambda_3=2.$

We have multiple eigenvalues if $K=0$ or $K=2$.

For $K=0$: $\lambda_1=\lambda_2=0$; $\lambda_3=2$. To find a basis for the eigenspace of $\lambda=0$:

$$\|A-0\cdot\mathbb{I}\|=\|A\| \Rightarrow \left\|\begin{matrix}1&0&1\\0&0&0\\1&0&1\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\mathbb{O} \Rightarrow \begin{cases}x+z=0\\ \forall y\end{cases} \Rightarrow \begin{cases}z=-x\\ \forall y\end{cases} : (x;y;-x).$$

Basis for the eigenspace: $\{(1;0;-1);(0;1;0)\}$

For $K=2$: $\lambda_1=0$; $\lambda_2=\lambda_3=2$. To find a basis for the eigenspace of $\lambda=2$:

$$\|A-2\mathbb{I}\|=\left\|\begin{matrix}-1&0&1\\0&0&0\\1&0&-1\end{matrix}\right\| \Rightarrow \left\|\begin{matrix}-1&0&1\\0&0&0\\1&0&-1\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\mathbb{O} \Rightarrow \begin{cases}x-z=0\\ \forall y\end{cases} \Rightarrow \begin{cases}z=x\\ \forall y\end{cases} : (x;y;x).$$

Basis for the eigenspace: $\{(1;0;1);(0;1;0)\}$.

IM4) $$\left\|\begin{matrix}1&2&-2&0\\3&-1&1&2\\3&m&8&K\end{matrix}\right\|\cdot\left\|\begin{matrix}x_1\\x_2\\x_3\\x_4\end{matrix}\right\|=\left\|\begin{matrix}1\\2\\1\end{matrix}\right\| \Rightarrow \left\|\begin{matrix}1&2&-2&0&\vdots&1\\3&-1&1&2&\vdots&2\\3&m&8&K&\vdots&1\end{matrix}\right\| \Rightarrow \left\|\begin{matrix}1&2&-2&0&\vdots&1\\0&-7&7&2&\vdots&-1\\0&m+1&7&K-2&\vdots&-1\end{matrix}\right\| \begin{pmatrix}R_2\leftarrow R_2-3R_1\\R_3\leftarrow R_3-R_2\end{pmatrix} \Rightarrow$$

$$\Rightarrow \left\|\begin{matrix}1&2&-2&0&\vdots&1\\0&-7&7&2&\vdots&-1\\0&m+8&0&K-4&\vdots&0\end{matrix}\right\| \quad (R_3\leftarrow R_3-R_2).$$

If $m=-8$ and $K=4$: $RANK(A)=RANK(A|Y)=2$: the system has ∞^2 Solutions;

If $m\neq-8$ OR $K\neq4$: $RANK(A)=RANK(A|Y)=3$: the system has ∞^1 Solutions.

The system has solutions $\forall m$ and $\forall K$.

IIM1) $f(x;y)=|xy|\cdot(x-y)$ is a continuous function $\forall (x;y)\in\mathbb{R}^2$. $f(0;0)=0$.

$$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to0}\frac{|(0+h)\cdot0|\cdot(0+h-0)-0}{h}=\lim_{h\to0}\frac{|0|\cdot h-0}{h}=\lim_{h\to0}\frac{0}{h}=0;$$

$$\frac{\partial f}{\partial y}(0;0)=\lim_{h\to0}\frac{|0\cdot(0+h)|\cdot(0-(0+h))-0}{h}=\lim_{h\to0}\frac{|0|\cdot(-h)-0}{h}=\lim_{h\to0}\frac{0}{h}=0.$$

$f(x;y)$ is differentiable ad $(0;0)$ if $\displaystyle\lim_{(x;y)\to(0;0)}\frac{|xy|\cdot(x-y)-0-(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}=0 \Rightarrow$

$$\Rightarrow \lim_{\rho\to0}\frac{\rho^2|\cos\vartheta\,\mathrm{sen}\,\vartheta|\cdot\rho(\cos\vartheta-\mathrm{sen}\,\vartheta)}{\rho}=\lim_{\rho\to0}\rho^2\cdot|\cos\vartheta\,\mathrm{sen}\,\vartheta|\cdot(\cos\vartheta-\mathrm{sen}\,\vartheta)=0.$$

The Convergence is uniform as: $|\rho^2|\cos\vartheta\,\mathrm{sen}\,\vartheta|(\cos\vartheta-\mathrm{sen}\,\vartheta)|\leq\rho^2\cdot1\cdot2=2\rho^2$.

So the function $f(x;y)$ is differentiable at $(0;0)$.

II M 2) $f(x;y)=e^{x-y}$ is a functione twice differentiable $\forall (x;y)\in\mathbb{R}^2$.

$\nabla f(x;y)=(e^{x-y};-e^{x-y})$. $H(f(x;y))=\begin{Vmatrix} e^{x-y} & -e^{x-y} \\ -e^{x-y} & e^{x-y}\end{Vmatrix}$.

$$\mathcal{D}_u f(x;y)=\nabla f(x;y)\cdot u=(e^{x-y};-e^{x-y})\cdot\left(\frac{\sqrt{3}}{2};\frac{1}{2}\right)=\frac{1}{2}e^{x-y}\cdot(\sqrt{3}-1).$$

$$\mathcal{D}^2_{u,v} f(x;y)=u\cdot H(f(x;y))\cdot v^T=\begin{Vmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{Vmatrix}\cdot\begin{Vmatrix} e^{x-y} & -e^{x-y} \\ -e^{x-y} & e^{x-y}\end{Vmatrix}\cdot\begin{Vmatrix}-\frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{Vmatrix}=\begin{Vmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\end{Vmatrix}\cdot\begin{Vmatrix} e^{x-y}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\right) \\ e^{x-y}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\end{Vmatrix}=$$

$$=e^{x-y}\left(-\frac{3}{4}-\frac{\sqrt{3}}{4}\right)+e^{x-y}\cdot\left(\frac{\sqrt{3}}{4}+\frac{1}{4}\right)=e^{x-y}\cdot\left(-\frac{1}{2}\right)=-2 \text{ if } e^{x-y}=4.$$

And so $\mathcal{D}_u f(x;y)=\frac{1}{2}e^{x-y}\cdot(\sqrt{3}-1)=\frac{1}{2}\cdot 4\cdot(\sqrt{3}-1)=2\cdot(\sqrt{3}-1)$.

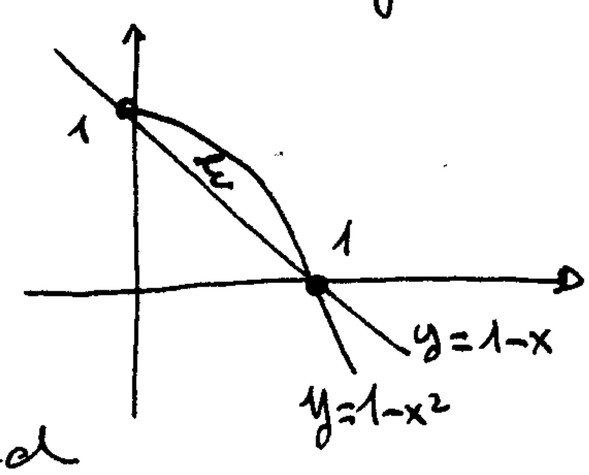
II M3) $\begin{cases} f(x;y;z)=xy-xz+y=0 \\ g(x;y;z)=xyz-xz+yz=0\end{cases}$; $P=(-1;1;0)\Rightarrow\begin{cases} f(-1;1;0)=-1-0+1=0 \\ g(-1;1;0)=0-0+0=0\end{cases}$.

$$\frac{\partial(f;g)}{\partial(x;y;z)}=\begin{Vmatrix} y-z & x+1 & -x \\ yz-z & xz+z & xy-x+y\end{Vmatrix};\ \frac{\partial(f;g)}{\partial(x;y;z)}(-1;1;0)=\begin{Vmatrix}1 & 0 & 1 \\ 0 & 0 & 1\end{Vmatrix}.$$

We can define an implicit function $y\to(x(y);z(y))$ as $\begin{vmatrix}1 & 1 \\ 0 & 1\end{vmatrix}=1\neq 0$.

$\frac{dx}{dy}=-\frac{\begin{vmatrix}0 & 1 \\ 0 & 1\end{vmatrix}}{\begin{vmatrix}1 & 1 \\ 0 & 1\end{vmatrix}}=0$; $\frac{dz}{dy}=-\frac{\begin{vmatrix}1 & 0 \\ 0 & 0\end{vmatrix}}{\begin{vmatrix}1 & 1 \\ 0 & 1\end{vmatrix}}=0$. There is no tangent vector at $y=1$.

II M4) $\begin{cases}\text{Max/min } f(x;y)=x^2-xy \\ \text{u.c. } \begin{cases}1-x\leq y \\ y\leq 1-x^2\end{cases}\end{cases}\Rightarrow\begin{cases}\text{Max/min } f(x;y)=x^2-xy \\ \text{u.c. } \begin{cases}1-x-y\leq 0 \\ y+x^2-1\leq 0\end{cases}\end{cases}$.



$f(x;y)$ is a continuous function; $\mathcal{E}$ is a bounded and closed set, constraints are qualified. There are solutions for Weierstrass. Theorem.

$$\Lambda=x^2-xy-\lambda_1(1-x-y)-\lambda_2(y+x^2-1).$$

MfEA4

If $\lambda_1 = \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = 2x - y = 0 \\ \Lambda'_y = -x = 0 \\ y \geq 1 - x \\ y \leq 1 - x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \geq 1 \\ 0 \leq 1 \end{cases} \Rightarrow (0;0) \notin \mathcal{E}.$$

If $\lambda_1 \neq 0$ and $\lambda_2 = 0$

$$\begin{cases} \Lambda'_x = 2x - y + \lambda_1 = 0 \\ \Lambda'_y = -x + \lambda_1 = 0 \\ y = 1 - x \\ y \leq 1 - x^2 \end{cases} \Rightarrow \begin{cases} x = \lambda_1 \\ y = 1 - \lambda_1 \\ 2\lambda_1 - 1 + \lambda_1 + \lambda_1 = 4\lambda_1 - 1 = 0 \\ y \leq 1 - x^2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{1}{4} > 0 \\ x = \frac{1}{4} \\ y = \frac{3}{4} \\ \frac{3}{4} \leq 1 - \frac{1}{16} = \frac{15}{16} : \text{true} \end{cases} \quad \left(\frac{1}{4}; \frac{3}{4}\right) \text{ MAX ??}$$

If $\lambda_1 = 0$ and $\lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2x - y - 2\lambda_2 x = 0 \\ \Lambda'_y = -x - \lambda_2 = 0 \\ y = 1 - x^2 \\ y \geq 1 - x \end{cases} \Rightarrow \begin{cases} x = -\lambda_2 \\ y = 1 - \lambda_2^2 \\ -2\lambda_2 - 1 + \lambda_2^2 + 2\lambda_2^2 = 3\lambda_2^2 - 2\lambda_2 - 1 = 0 \\ y \geq 1 - x \end{cases} \Rightarrow \begin{cases} \lambda_2 = 1 \\ x = -1 \\ y = 0 \\ 0 \geq 2 \end{cases} \cup \begin{cases} \lambda_2 = -\frac{1}{3} < 0 \left(\frac{1}{3}; \frac{8}{9}\right) \\ x = \frac{1}{3} \\ y = \frac{8}{9} \\ \frac{8}{9} \geq \frac{2}{3} : \text{true} \end{cases} \text{ min ??}$$

$$\lambda_2 = \frac{1 \pm \sqrt{1+3}}{3} = \frac{1 \pm 2}{3}$$

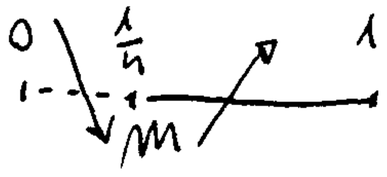
$(-1;0) \notin \mathcal{E}$

If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = 2x - y + \lambda_1 - 2\lambda_2 x = 0 \\ \Lambda'_y = -x + \lambda_1 - \lambda_2 = 0 \\ y = 1 - x \\ y = 1 - x^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \cup \begin{cases} x = 1 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = 1 \\ \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = +1 > 0 \\ \lambda_2 = +1 > 0 \end{cases} \cup \begin{cases} x = 1 \\ y = 0 \\ \lambda_1 - 2\lambda_2 = -2 \\ \lambda_1 - \lambda_2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_1 = 4 > 0 \\ \lambda_2 = 3 > 0 \end{cases}$$

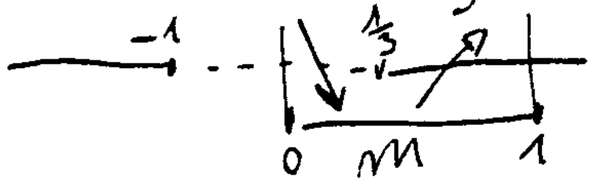
$(0;1)$ MAX ?? $\qquad$ $(1;0)$ MAX ??

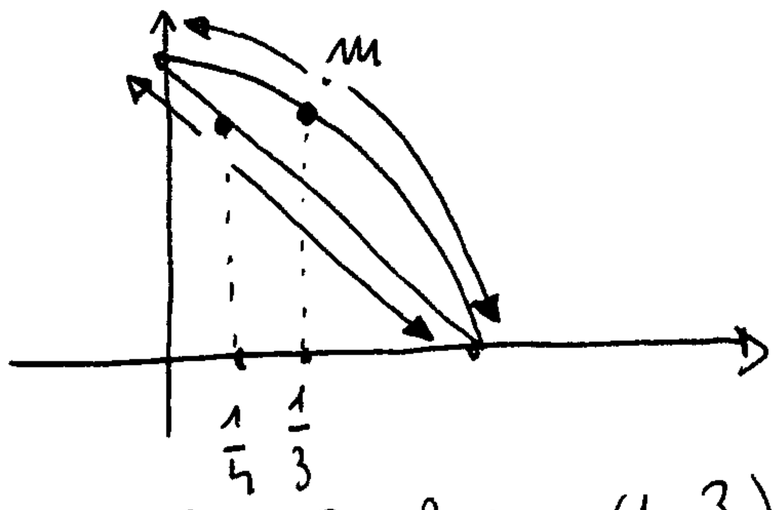
$f(x; 1-x) = 2x^2 - x$; $f'(x) = 4x - 1 \geq 0$: $x \geq \frac{1}{4}$



$f(x; 1-x^2) = x^3 + x^2 - x$; $f'(x) = 3x^2 + 2x - 1 \geq 0$

$x \leq -1 \cup x \geq \frac{1}{3}$





$\left(\frac{1}{3}; \frac{8}{9}\right)$ P. of Min; $(0;1)$ and $(1;0)$ P. of Max. $\left(\frac{1}{4}; \frac{3}{4}\right)$ Nothing.