TASK MATHEMATICS for ECONOMIC APPLICATIONS 17/2/2015 $\boxed{\text{MfEA 1}}$

IM1) $e^{\log 2+\frac{5}{3}\pi i}=e^{\log 2}\cdot e^{\frac{5}{3}\pi i}=2\cdot\left(\cos\frac{5}{3}\pi+i\,\text{sen}\,\frac{5}{3}\pi\right)=2\cdot\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=1-\sqrt{3}i.$

$\sqrt[3]{z}=\sqrt[3]{2}\cdot\left(\cos\left(\frac{5}{9}\pi+k\cdot\frac{2\pi}{3}\right)+i\,\text{sen}\left(\frac{5}{9}\pi+k\cdot\frac{2\pi}{3}\right)\right);\ 0\le k\le 2.$

$k=0:\sqrt[3]{2}\left(\cos\frac{5}{9}\pi+i\,\text{sen}\,\frac{5}{9}\pi\right);\ k=1:\sqrt[3]{2}\left(\cos\frac{11}{9}\pi+i\,\text{sen}\,\frac{11}{9}\pi\right);\ k=2:\sqrt[3]{2}\cdot\left(\cos\frac{17}{9}\pi+i\,\text{sen}\,\frac{17}{9}\pi\right).$

IM2) $f(x_1;x_2;x_3;x_4)=(x_1+x_4;2x_1+x_2+x_3;x_1+x_2+x_3-x_4).$

$$f(x)=\begin{Vmatrix}1&0&0&1\\2&1&1&0\\1&1&1&-1\end{Vmatrix}\cdot\begin{Vmatrix}x_1\\x_2\\x_3\\x_4\end{Vmatrix}\Rightarrow\begin{Vmatrix}1&0&0&1\\2&1&1&0\\1&1&1&-1\end{Vmatrix}\underset{\substack{R_2\leftarrow R_2-2R_1\\R_3\leftarrow R_3-R_1}}{\Rightarrow}\begin{Vmatrix}1&0&0&1\\0&1&1&-2\\0&1&1&-2\end{Vmatrix}\underset{R_3\leftarrow R_3-R_2}{\Rightarrow}\begin{Vmatrix}1&0&0&1\\0&1&1&-2\\0&0&0&0\end{Vmatrix}$$

RANK$(A)=2\Rightarrow$ Dim (Imm) $=2$ and Dim (Ker) $=4-2=2.$

Basis for Kernel: $A\circ X=\underline{0}\Rightarrow\begin{Vmatrix}1&0&0&1\\0&1&1&-2\end{Vmatrix}\cdot\begin{Vmatrix}x_1\\x_2\\x_3\\x_4\end{Vmatrix}=\begin{Vmatrix}0\\0\end{Vmatrix}\Rightarrow\begin{cases}x_1+x_4=0\\x_2+x_3-2x_4=0\end{cases}\Rightarrow$

$\Rightarrow\begin{cases}x_4=-x_1\\x_2+x_3+2x_1=0\end{cases}\Rightarrow\begin{cases}x_4=-x_1\\x_3=-2x_1-x_2\end{cases}\Rightarrow(x_1;x_2;-2x_1-x_2;-x_1)\rightarrow(1;0;-2;-1),\ (0;1;-1;0)$: Basis for KERNEL.

Basis for Image:
$$\left\|\begin{array}{cccc:c}1&0&0&1&y_1\\2&1&1&0&y_2\\1&1&1&-1&y_3\end{array}\right\|\Rightarrow\left\|\begin{array}{cccc:c}1&0&0&1&y_1\\0&1&1&-2&y_2-2y_1\\0&1&1&-2&y_3-y_1\end{array}\right\|\Rightarrow\left\|\begin{array}{cccc:c}1&0&0&1&y_1\\0&1&1&-2&y_2-2y_1\\0&0&0&0&y_3-y_2+y_1\end{array}\right\|$$

We use the same elementary operations.

So we need: $y_3-y_2+y_1=0\Rightarrow y_3=y_2-y_1\Rightarrow(y_1;y_2;y_2-y_1)\rightarrow(1;0;-1),\ (0;1;1)$: Basis for IMAGE.

IM3) $A=\begin{Vmatrix}1&0&k\\0&1&0\\k&0&1\end{Vmatrix}\Rightarrow\begin{vmatrix}1-\lambda&0&k\\0&1-\lambda&0\\k&0&1-\lambda\end{vmatrix}=(1-\lambda)\cdot\left((1-\lambda)^2-k^2\right)=(1-\lambda)(\lambda^2-2\lambda+1-k^2)=0.$

If $\lambda=1$ is a multiple eigenvalue: $1-2+1-k^2\Rightarrow k^2=0\Rightarrow k=0.\ A=\begin{Vmatrix}1&0&0\\0&1&0\\0&0&1\end{Vmatrix}=\mathbb{I}_3.$

$P_3(\lambda)=(1-\lambda)\cdot(\lambda^2-2\lambda+1)=-(\lambda-1)^3\Rightarrow\lambda_1=\lambda_2=\lambda_3=1.$

We must find three orthogonal eigenvectors for $\lambda=1$ and $k=0$.

Since the matrix $A = \mathbb{I}_3$, the orthogonal matrix is the matrix A itself, as $\mathbb{I}_3 \cdot A \cdot \mathbb{I}_3 = \mathbb{I}_3$ and $\mathbb{I}_3 = D$: diagonal matrix.

IM4) $A = \begin{Vmatrix} 3 & 0 & 5 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{Vmatrix}$. $|A| = 1 \cdot \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 1 \cdot (6-5) = 1.$

$$\begin{Vmatrix} 3 & 0 & 5 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{Vmatrix} \overset{Adj}{\Longrightarrow} \begin{Vmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -5 & 0 & 3 \end{Vmatrix} \overset{T}{\Longrightarrow} \begin{Vmatrix} 2 & 0 & -5 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{Vmatrix} = A^{-1}.$$

$$\begin{Vmatrix} 3 & 0 & 5 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 2 \\ -1 \\ 3 \end{Vmatrix} \Rightarrow \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 3 & 0 & 5 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} 2 \\ -1 \\ 3 \end{Vmatrix} = \begin{Vmatrix} 2 & 0 & -5 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{Vmatrix} \cdot \begin{Vmatrix} 2 \\ -1 \\ 3 \end{Vmatrix} = \begin{Vmatrix} -11 \\ -1 \\ 7 \end{Vmatrix}.$$

IIM1) $f(x;y) = x^2 - xy^2$: differentiable function of the second order.

$\nabla f(x;y) = (2x - y^2; -2xy)$. $H(x;y) = \begin{Vmatrix} 2 & -2y \\ -2y & -2x \end{Vmatrix}$. $u = \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$; $v = \left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right)$.

$$D_u f(x;y) = \nabla f(x;y) \cdot u = (2x - y^2; -2xy) \cdot \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(2x - y^2 - 2xy\right).$$

$$D^2_{u,v} f(x;y) = u \cdot H \cdot v^T = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} 2 & -2y \\ -2y & -2x \end{Vmatrix} \cdot \begin{Vmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{Vmatrix} = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} \frac{1}{\sqrt{2}}(2+2y) \\ \frac{1}{\sqrt{2}}(-2y+2x) \end{Vmatrix} =$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot (2 + 2y - 2y + 2x) = \frac{1}{2}(2 + 2x) = 1 + x.$$

So we get the system: $\begin{cases} \frac{1}{\sqrt{2}}(2x - y^2 - 2xy) = \sqrt{2} \\ 1 + x = 2 \end{cases} \Rightarrow \begin{cases} 2x - y^2 - 2xy = 2 \\ 1 + x = 2 \end{cases} \Rightarrow \begin{cases} y^2 + 2y = 0 \\ x = 1 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = 1 \\ y(y+2) = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$ and $\begin{cases} x = 1 \\ y = -2 \end{cases}$. We find two points: $P_1 = (1;0)$ and $P_2 = (1;-2)$.

IIM2) $f(x;y) = y \log x - x e^y + x = 0$; $f(1;0) = 0 - 1 + 1 = 0$.

$\nabla f(x;y) = \left(y \cdot \frac{1}{x} - e^y + 1; \log x - x e^y\right)$; $\nabla f(1;0) = (0 - 1 + 1; 0 - 1) = (0; -1)$.

$f'_y(1;0) \neq 0$ so exists $x: x \to y(x)$ implicit function.

$$y'(1) = -\frac{f'_x(1;0)}{f'_y(1;0)} = -\frac{0}{-1} = 0.$$

$$H\left(f(x;y)\right) = \begin{Vmatrix} -y\cdot\frac{1}{x^2} & \frac{1}{x}-e^y \\ \frac{1}{x}-e^y & -xe^y \end{Vmatrix}; \quad H\left(f(1;0)\right) = \begin{Vmatrix} 0 & 0 \\ 0 & -1 \end{Vmatrix}.$$

$$y''(1) = -\frac{f''_{xx}(1;0) + 2f''_{xy}(1;0)\cdot y' + f''_{yy}(1;0)\cdot(y')^2}{f'_y(1;0)} = -\frac{0+2\cdot 0\cdot 0+(-1)\cdot(0)^2}{-1} = 0.$$

II M 3) $f(x;y) = x^2 - xy^2 + kxy.$

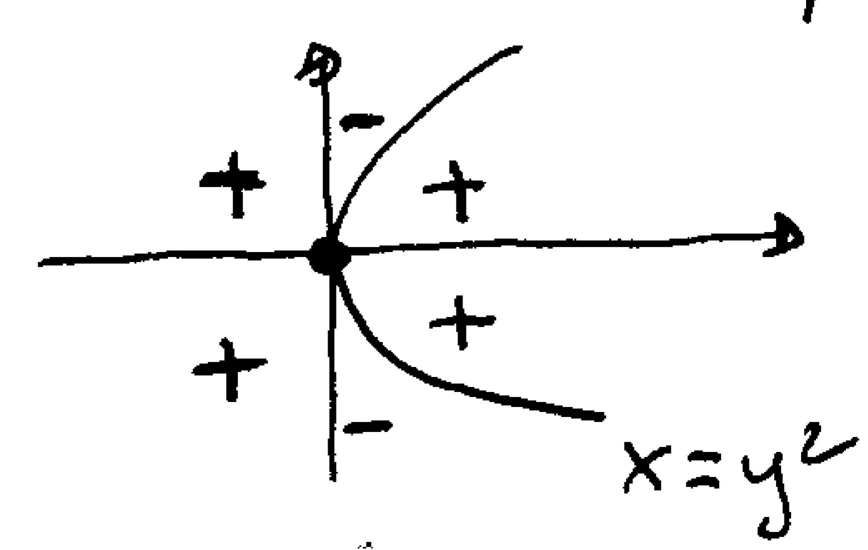
$$\nabla f(x;y) = (0;0) \Rightarrow \begin{cases} f'_x = 2x - y^2 + ky = 0 \\ f'_y = -2xy + kx = 0 \end{cases} \Rightarrow \begin{cases} x(k-2y) = 0 \\ 2x - y^2 + ky = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y(k-y) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \cup \begin{cases} x=0 \\ y=k \end{cases} \text{ or } \begin{cases} y=\frac{k}{2} \\ 2x-\frac{k^2}{4}+\frac{k^2}{2} = 2x+\frac{k^2}{4} = 0 \end{cases} \Rightarrow \begin{cases} y=\frac{k}{2} \\ x=-\frac{k^2}{8} \end{cases}.$$

We have three stationary points: $P_1=(0;0)$; $P_2=(0;k)$; $P_3=\left(-\frac{k^2}{8};\frac{k}{2}\right)$.

$$H(x;y) = \begin{Vmatrix} 2 & k-2y \\ k-2y & -2x \end{Vmatrix}. \quad H(0;0) = \begin{Vmatrix} 2 & k \\ k & 0 \end{Vmatrix}.$$ For $k\neq 0$: $|H_2| = -k^2<0$: Saddle point.

For $k=0$: $f(x;y) = x^2-xy^2 = x\cdot(x-y^2)$; $f(0;0)=0$; $f(x;y)\gtrless 0$:

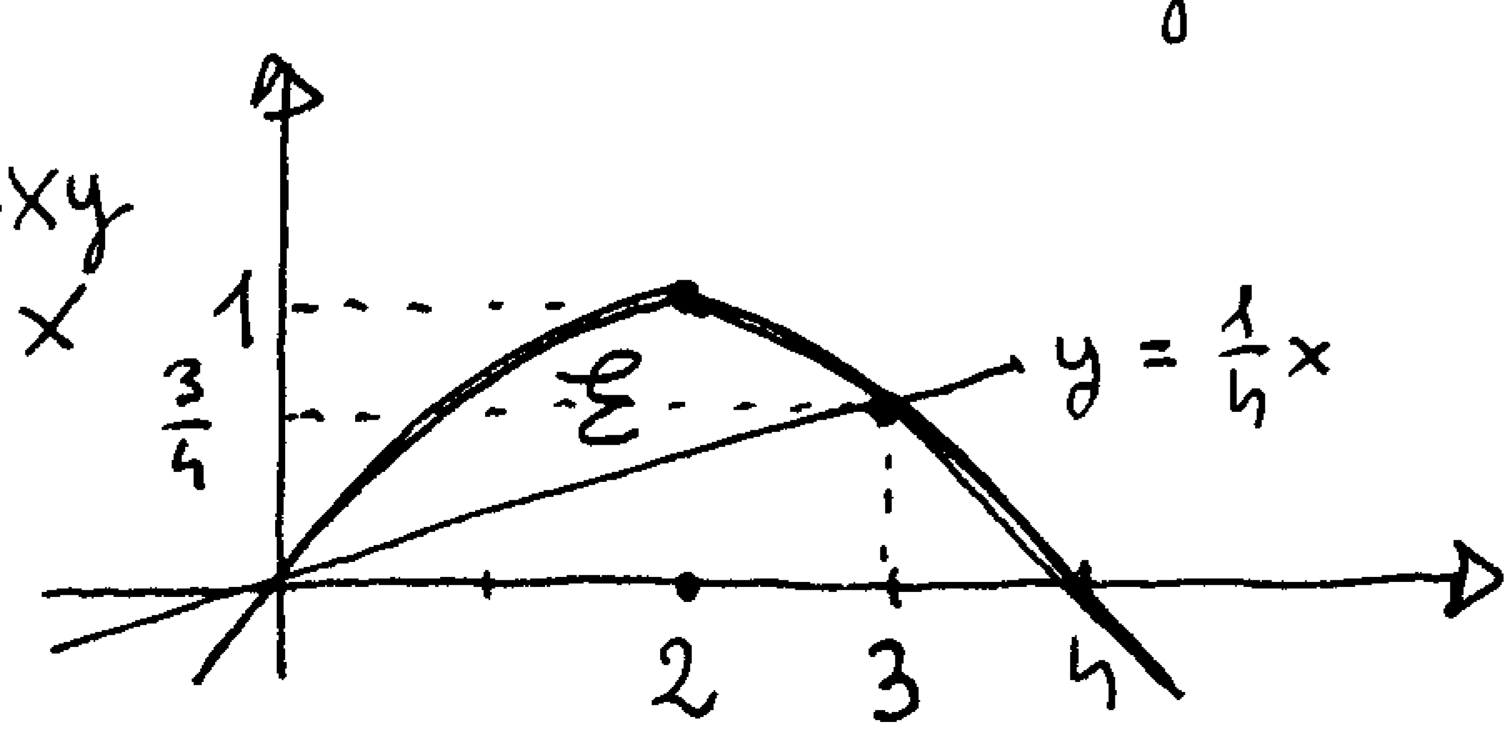


For $k=0$ $P_1=(0;0)$ is a saddle point.

$$H(0;k) = \begin{Vmatrix} 2 & -k \\ -k & 0 \end{Vmatrix}.$$ For $k\neq 0$: $|H_2| = -k^2<0$: Saddle Point. For $k=0$: already studied.

$$H\left(-\frac{k^2}{8};\frac{k}{2}\right) = \begin{Vmatrix} 2 & 0 \\ 0 & \frac{k^2}{4} \end{Vmatrix}.$$ For $k\neq 0$: $\begin{cases} |H_1|>0 \\ |H_2|>0 \end{cases}$: Minimum Point. For $k=0$: already studied.

II M4) $$\begin{cases} \text{Max/min } f(x;y) = x\cdot y \\ \text{u.c.} \begin{cases} x^2-4x+4y \le 0 \\ x-4y \le 0 \end{cases} \end{cases} \Rightarrow \begin{cases} \text{Max/min } f(x;y) = xy \\ \text{u.c.} \begin{cases} y \le -\frac{1}{4}x^2+x \\ y \ge \frac{1}{4}x \end{cases} \end{cases}$$



MfEA4

$\mathcal{E}$ is a bounded and closed set. The constraints are qualified.

$f(x;y)$ is a continuous function: there are Maximum and minimum points from Weierstrass Theorem

$$\Lambda = xy - \lambda_1(x^2 - 4x + 4y) - \lambda_2(x - 4y).$$

<u>If $\lambda_1 = \lambda_2 = 0$</u>

$$\begin{cases} \Lambda'_x = y = 0 \\ \Lambda'_y = x = 0 \\ x^2 - 4x + 4y \leq 0 \\ x - 4y \leq 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \leq 0 \\ 0 \leq 0. \end{cases} \quad H(x;y) = H(0;0) = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}; \ |H_2| = -1 < 0: \text{Saddle point for free Max/min.}$$

<u>If $\lambda_1 \neq 0; \lambda_2 = 0$</u>

$$\begin{cases} \Lambda'_x = y - 2\lambda_1 x + 4\lambda_1 = 0 \\ \Lambda'_y = x - 4\lambda_1 = 0 \\ y = x - \frac{x^2}{4} \\ y \geq \frac{1}{4}x \end{cases} \Rightarrow \begin{cases} x = 4\lambda_1 \\ y - 8\lambda_1^2 + 4\lambda_1 = 0 \\ y = x - \frac{x^2}{4} \\ y \geq \frac{1}{4}x \end{cases} \Rightarrow \begin{cases} x = 4\lambda_1 \\ y = 8\lambda_1^2 - 4\lambda_1 \\ 8\lambda_1^2 - 4\lambda_1 = 4\lambda_1 - 4\lambda_1^2 \\ y \geq \frac{1}{4}x \end{cases} \Rightarrow$$

$$\Rightarrow 12\lambda_1^2 + 8\lambda_1 = 4\lambda_1(3\lambda_1 - 2) = 0 \Rightarrow \lambda_1 = 0 \text{ and } \lambda_1 = \frac{2}{3}.$$

$$\begin{cases} \lambda_1 = 0 \\ x = 0 \\ y = 0 \\ 0 \geq 0 \end{cases} \text{already studied}; \quad \begin{cases} \lambda_1 = \frac{2}{3} > 0 \\ x = \frac{8}{3} \\ y = \frac{8}{3} - \frac{1}{4} \cdot \frac{64}{9} = \frac{96 - 64}{36} = \frac{32}{36} = \frac{8}{9} \\ \frac{8}{9} \geq \frac{8}{12} \text{ satisfied.} \end{cases} : \left(\frac{8}{3}; \frac{8}{9}\right) \text{ MAX ??}$$

<u>If $\lambda_1 = 0; \lambda_2 \neq 0$</u>

$$\begin{cases} \Lambda'_x = y - \lambda_2 = 0 \\ \Lambda'_y = x + 4\lambda_2 = 0 \\ y = \frac{1}{4}x \\ y \leq x - \frac{x^2}{4} \end{cases} \Rightarrow \begin{cases} x = -4\lambda_2 \\ y = \lambda_2 \\ \lambda_2 = \frac{1}{4} \cdot (-4\lambda_2) \\ y \leq x - \frac{x^2}{4} \end{cases} \Rightarrow \begin{cases} 2\lambda_2 = 0 \Rightarrow \lambda_2 = 0 \\ x = 0 \\ y = 0 \\ 0 \leq 0 \end{cases} \quad \text{already studied.}$$

MfEA5

<u>If $\lambda_1 \neq 0;\ \lambda_2 \neq 0$</u>

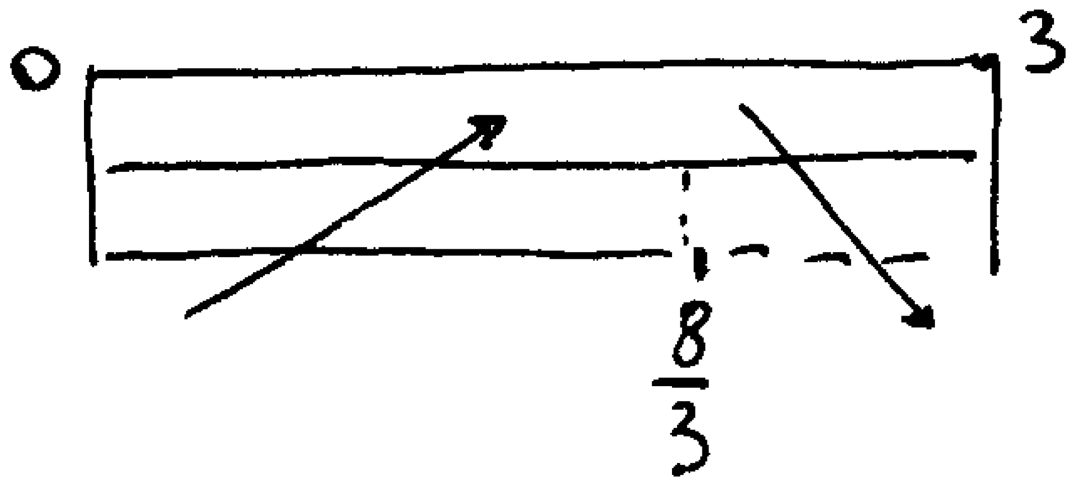
$$\begin{cases} \Lambda'_x = y - 2\lambda_1 x + 4\lambda_1 - \lambda_2 = 0 \\ \Lambda'_y = x - 4\lambda_1 + 4\lambda_2 = 0 \\ \begin{cases} y = x - \frac{x^2}{4} \\ y = \frac{1}{4}x \end{cases} \end{cases} \Rightarrow \begin{cases} \frac{1}{4}x = x - \frac{x^2}{4} \\ y = \frac{1}{4}x \end{cases} \Rightarrow \begin{cases} x^2 - 3x = 0 \\ y = \frac{1}{4}x \end{cases} \Rightarrow \begin{cases} x(x-3) = 0 \\ y = \frac{1}{4}x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ and } \begin{cases} x = 3 \\ y = \frac{3}{4} \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \\ 4\lambda_1 - \lambda_2 = 0 \\ -4\lambda_1 + 4\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \text{ already studied.}$$

$$\begin{cases} x = 3 \\ y = \frac{3}{4} \\ \frac{3}{4} - 6\lambda_1 + 4\lambda_1 - \lambda_2 = 0 \\ 3 - 4\lambda_1 + 4\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = \frac{3}{4} \\ 2\lambda_1 + \lambda_2 = \frac{3}{4} \\ 4\lambda_1 - 4\lambda_2 = 3 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = \frac{3}{4} \\ \lambda_2 = \frac{3}{4} - 2\lambda_1 \\ 4\lambda_1 - 3 + 8\lambda_1 = 12\lambda_1 - 3 = 3 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = \frac{3}{4} \\ \lambda_1 = \frac{1}{2} > 0 \\ \lambda_2 = -\frac{1}{4} < 0 \end{cases} \text{ } \underline{\text{Nothing}}$$

For $y = x - \frac{x^2}{4}$: $f(x) = x\left(x - \frac{x^2}{4}\right) = x^2 - \frac{x^3}{4}$; $f'(x) = 2x - \frac{3}{4}x^2 = x\left(2 - \frac{3}{4}x\right) \geq 0 \Rightarrow$

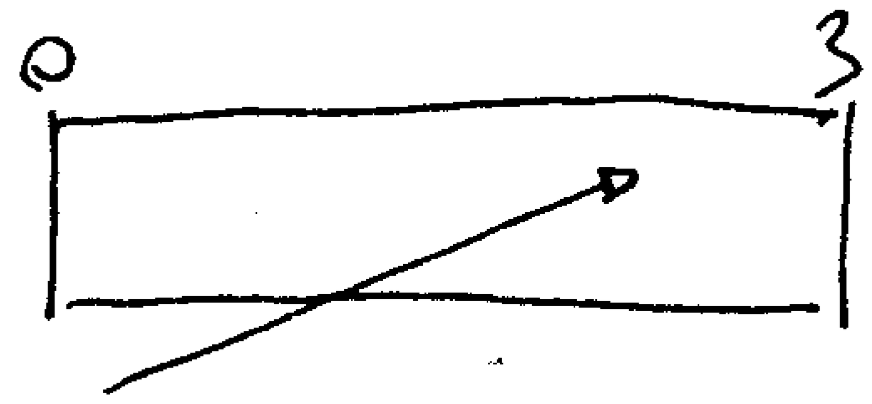
$$\Rightarrow \begin{cases} x \geq 0 \\ 2 - \frac{3}{4}x \geq 0 \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x \leq \frac{8}{3} \end{cases}$$



$x \geq 0$
$x \leq \frac{8}{3}$

Maximum at $x = \frac{8}{3}$ and $y = \frac{8}{9}$.

For $y = \frac{1}{4}x$: $f(x) = x \cdot \frac{1}{4}x = \frac{1}{4}x^2$; $f'(x) = \frac{1}{2}x \geq 0 \Rightarrow x \geq 0$



$x \geq 0$

$\left(\frac{8}{3}; \frac{8}{9}\right)$ is the absolute Maximum point with $f\left(\frac{8}{3}; \frac{8}{9}\right) = \frac{64}{27}$;
$(0;0)$ is not a Saddle point but it is the minimum point, with $f(0;0) = 0$.