TASK MATHEMATICS for ECONOMIC APPLICATIONS 28/3/2015 $\boxed{\text{MfEA1}}$

IM1) $5\cdot\left(\frac{1+i}{1+3i}-\frac{2-2i}{3+i}\right)=5\cdot\left(\frac{(1+i)(1-3i)}{(1+3i)(1-3i)}-\frac{(2-2i)(3-i)}{(3+i)(3-i)}\right)=5\cdot\left(\frac{4-2i}{10}-\frac{4-8i}{10}\right)=$

$=5\left(\frac{6i}{10}\right)=3i=3\left(\cos\frac{\pi}{2}+i\,\text{sen}\,\frac{\pi}{2}\right).$

$\sqrt{3i}=\sqrt{3}\cdot\left(\cos\left(\frac{\pi}{4}+k\cdot\frac{2\pi}{2}\right)+i\,\text{sen}\left(\frac{\pi}{4}+k\,\frac{2\pi}{2}\right)\right);\ 0\leq k\leq 1.$

For $k=0$: $\sqrt{3}\left(\cos\frac{\pi}{4}+i\,\text{sen}\,\frac{\pi}{4}\right)=\sqrt{3}\left(\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}\right)$;

For $k=1$: $\sqrt{3}\left(\cos\frac{5}{4}\pi+i\,\text{sen}\,\frac{5}{4}\pi\right)=\sqrt{3}\left(-\frac{1}{\sqrt{2}}-i\,\frac{1}{\sqrt{2}}\right).$

IM2) $A=\left\|\begin{matrix}1&-1&2&-2\\-2&2&m&k\end{matrix}\right\|\xrightarrow[R_2\leftarrow R_2+2R_1]{}\left\|\begin{matrix}1&-1&2&-2\\0&0&m+4&k-4\end{matrix}\right\|$. Since $\text{Dim}(\text{Imm})=1\Rightarrow \text{RANK}(A)=1$

$\Rightarrow \text{RANK}(A)=1$ iff $\begin{cases}m+4=0\\k-4=0\end{cases}\Rightarrow\begin{cases}m=-4\\k=4\end{cases}$: $A=\left\|\begin{matrix}1&-1&2&-2\\-2&2&-4&4\end{matrix}\right\|.$

Basis for Kernel: $\text{Dim}(\text{Ker})=4-\text{RANK}(A)=4-1=3.$

$\left\|\begin{matrix}1&-1&2&-2\\-2&2&-4&4\end{matrix}\right\|\cdot(x_1;x_2;x_3;x_4)^T=\underline{0}\Rightarrow x_1-x_2+2x_3-2x_4=0\Rightarrow x_2=x_1+2x_3-2x_4$

$X=(x_1;x_1+2x_3-2x_4;x_3;x_4)\Rightarrow\begin{cases}\text{if } x_1=1;x_3=0;x_4=0: & (1;1;0;0)\\ \text{if } x_1=0;x_3=1;x_4=0: & (0;2;1;0)\text{ : BASIS.}\\ \text{if } x_1=0;x_3=0;x_4=1: & (0;-2;0;1)\end{cases}$

Basis for Image: $\left\|\begin{matrix}1&-1&2&-2&\vdots&y_1\\-2&2&-4&4&\vdots&y_2\end{matrix}\right\|\xrightarrow[R_2\leftarrow R_2+2R_1]{}\left\|\begin{matrix}1&-1&2&-2&\vdots&y_1\\0&0&0&0&\vdots&y_2+2y_1\end{matrix}\right\|\Rightarrow$

$\Rightarrow y_2+2y_1=0\Rightarrow y_2=-2y_1$. $Y=(y_1;-2y_1)$. If $y_1=0$: $(1;-2)$: Basis.

IM3) $A=\left\|\begin{matrix}0&1&0\\1&0&1\\0&k&1\end{matrix}\right\|\Rightarrow\left|\begin{matrix}-\lambda&1&0\\1&-\lambda&1\\0&k&1-\lambda\end{matrix}\right|=\left|\begin{matrix}-\lambda&1&0\\0&-\lambda&1\\\lambda-1&k&1-\lambda\end{matrix}\right|=(-\lambda)(\lambda^2-\lambda-k)+(\lambda-1)\cdot 1\Rightarrow$

$\Rightarrow$ For $\lambda=-1$: $(1)\cdot(1+1-k)+(-2)\cdot 1=2-k-2=0\Rightarrow k=0.$

Since $k=0$: $-\lambda^3+\lambda^2+\lambda-1=-(\lambda^3-\lambda^2-\lambda+1)=0$. $\lambda=-1$: $\begin{array}{r|rrr|r} & 1 & -1 & -1 & 1\\ -1 & & -1 & 2 & -1\\ \hline & 1 & -2 & 1 & 0\end{array}\Rightarrow$

$\Rightarrow \lambda^3-\lambda^2-\lambda+1=(\lambda+1)(\lambda^2-2\lambda+1)=(\lambda+1)(\lambda-1)^2=0.$

So $\lambda_1 = -1$; $\lambda_2 = \lambda_3 = 1$. $\lambda = 1$ Multiple Eigenvalue.

For $k=0$ and $\lambda = 1$: $\left\|A - 1\cdot \mathbb{I}\right\| = \begin{Vmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{Vmatrix}$. Since $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \neq 0$, RANK $(A - 1\cdot\mathbb{I}) = 2 \Rightarrow$

$\Rightarrow m_1^g = 3 - 2 = 1 < 2 = m_1^a$. The Matrix for $k=0$ is not a diagonalizable one.

IM4) For the similarity: $A\cdot P = P\cdot B \Rightarrow \begin{Vmatrix} 1 & 2 \\ 1 & 3 \end{Vmatrix} \cdot \begin{Vmatrix} 2 & 1 \\ 3 & 2 \end{Vmatrix} = \begin{Vmatrix} 2 & 1 \\ 3 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} x & 1 \\ 1 & y \end{Vmatrix} \Rightarrow$

$\Rightarrow \begin{Vmatrix} 8 & 5 \\ 11 & 7 \end{Vmatrix} = \begin{Vmatrix} 2x+1 & 2+y \\ 3x+2 & 3+2y \end{Vmatrix} \Rightarrow \begin{cases} 2x+1 = 8 \\ 3x+2 = 11 \end{cases}$ and $\begin{cases} 2+y = 5 \\ 3+2y = 7 \end{cases} \Rightarrow \begin{cases} x = \frac{7}{2} \\ x = 3 \end{cases}$ and $\begin{cases} y = 3 \\ y = 2 \end{cases}$.

So the system is impossible and has no solutions. Values x and y do not exist.

IIM1) $f(x;y) = e^{ax+y} + e^{x-by}$ is a twice differentiable function.

$\nabla f(x;y) = \left(a e^{ax+y} + e^{x-by}\,;\, e^{ax+y} - b e^{x-by}\right)$. $\nabla f(0;0) = (a+1\,;\,1-b)$.

$H(f(x;y)) = \begin{Vmatrix} a^2 e^{ax+y} + e^{x-by} & a e^{ax+y} - b e^{x-by} \\ a e^{ax+y} - b e^{x-by} & e^{ax+y} + b^2 e^{x-by} \end{Vmatrix}$. $H(f(0;0)) = \begin{Vmatrix} a^2+1 & a-b \\ a-b & 1+b^2 \end{Vmatrix}$.

$\mathcal{D}_v f(0;0) = \nabla f(0;0)\cdot v = (a+1; 1-b)\cdot\left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\cdot(a+1-1+b) = 0 \Rightarrow a+b = 0 \Rightarrow \underline{b=-a}$;

$\mathcal{D}^2_{v,-v} f(0;0) = \begin{Vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} a^2+1 & a-b \\ a-b & 1+b^2 \end{Vmatrix} \cdot \begin{Vmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Vmatrix} = \begin{Vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} \frac{1}{\sqrt{2}}(-a^2-1+a-b) \\ \frac{1}{\sqrt{2}}(-a+b+1+b^2) \end{Vmatrix} =$

$= \frac{1}{2}\left(-a^2-1+a-b+a-b-1-b^2\right) = \frac{1}{2}\left(2a-2b-2-a^2-b^2\right) = 0$. Since $b=-a$:

$\Rightarrow 2a+2a-2-a^2-a^2 = -2(a^2-2a+1) = -2\cdot(a-1)^2 = 0 \Rightarrow a = 1$ and $b = -1$.

IIM2) $f(x;y) = y e^{y-x} + e^{x-y} = 2$; $f(1;1) = 1\cdot e^0 + e^0 = 1+1 = 2$.

$\nabla f(x;y) = \left(-y e^{y-x} + e^{x-y}\,;\, e^{y-x} + y e^{y-x} - e^{x-y}\right)$; $\nabla f(1;1) = (0\,;\,1)$. $f'_y(1;1) \neq 0$.

$y'(1) = -\frac{f'_x}{f'_y} = -\frac{0}{1} = 0$: $x=1$ is a stationary point for $y(x)$.

$H(f(x;y)) = \begin{Vmatrix} y e^{y-x} + e^{x-y} & -e^{y-x} - y e^{y-x} - e^{x-y} \\ -e^{y-x} - y e^{y-x} - e^{x-y} & e^{y-x} + e^{y-x} + y e^{y-x} + e^{x-y} \end{Vmatrix}$. $H(f(1;1)) = \begin{Vmatrix} 2 & -3 \\ -3 & 4 \end{Vmatrix}$.

$$y''(1) = -\frac{f''_{xx} + 2f''_{xy}\cdot y' + f''_{yy}\cdot(y')^2}{f'_y} = -\frac{2+2\cdot(-3)\cdot 0 + 4\cdot 0^2}{1} = -2 < 0$$

Since $y'(1) = 0$ and $y''(1) = -2 < 0 \Rightarrow x=1$ is a Maximum point.

II M3) $g: (x;y) \to (x-y; x+y) = (v;w)$; $f:(v;w) \to (vw; v^2-w^2) = (f_1; f_2)$

$$J(g) = \begin{Vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{Vmatrix} = \begin{Vmatrix} 1 & -1 \\ 1 & 1 \end{Vmatrix} ; \quad J(f) = \begin{Vmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \end{Vmatrix} = \begin{Vmatrix} w & v \\ 2v & -2w \end{Vmatrix}.$$

$$J(f\circ g) = J(f)\cdot J(g) = \begin{Vmatrix} w & v \\ 2v & -2w \end{Vmatrix} \cdot \begin{Vmatrix} 1 & -1 \\ 1 & 1 \end{Vmatrix} = \begin{Vmatrix} w+v & -w+v \\ 2v-2w & -2v-2w \end{Vmatrix}.$$

Substituting: $\begin{Vmatrix} x+y+x-y & -x-y+x-y \\ 2x-2y-2x-2y & -2x+2y-2x-2y \end{Vmatrix} = \begin{Vmatrix} 2x & -2y \\ -4y & -4x \end{Vmatrix}.$

$|J(f\circ g)| = -8x^2 - 8y^2 = F(x;y).$

$$\begin{cases} F'_x = -16x = 0 \\ F'_y = -16y = 0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} ; \quad H = \begin{Vmatrix} -16 & 0 \\ 0 & -16 \end{Vmatrix} \Rightarrow \begin{cases} F''_{xx} < 0 ;\ F''_{yy} < 0 \\ |H_2| = (-16)^2 > 0 \end{cases} \Rightarrow (0;0) \text{ is a Maximum Point.}$$

II M4) $\begin{cases} \text{Max/min } f(x;y) = x^2+y \\ \text{u.c.}: x^2+y^2 = 1 \end{cases}$. $\Lambda = x^2+y-\lambda(x^2+y^2)$

$$\begin{cases} \Lambda'_x = 2x - 2\lambda x = 2x(1-\lambda) \\ \Lambda'_y = 1 - 2\lambda y \\ x^2+y^2 = 1 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=\pm 1 \\ \lambda = \frac{1}{2y} \end{cases} \Rightarrow \begin{cases} (0;1;\frac{1}{2}) \\ (0;-1;-\frac{1}{2}) \end{cases} \cup \begin{cases} \lambda = 1 \\ y = \frac{1}{2} \\ x = \pm\frac{\sqrt{3}}{2} \end{cases} \Rightarrow \begin{cases} (\frac{\sqrt{3}}{2}; \frac{1}{2}; 1) \\ (-\frac{\sqrt{3}}{2}; \frac{1}{2}; 1) \end{cases}.$$

$$\overline{H} = \begin{Vmatrix} 0 & 2x & 2y \\ 2x & 2-2\lambda & 0 \\ 2y & 0 & -2\lambda \end{Vmatrix}. \quad |\overline{H}(0;1;\tfrac{1}{2})| = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix} = -4 < 0: \underline{\text{MIN}}. ; \quad |\overline{H}(0;-1;-\tfrac{1}{2})| = \begin{vmatrix} 0 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{vmatrix} = -12 < 0: \underline{\text{MIN}}$$

$$|\overline{H}(\tfrac{\sqrt{3}}{2}; \tfrac{1}{2}; 1)| = \begin{vmatrix} 0 & \sqrt{3} & 1 \\ \sqrt{3} & 0 & 0 \\ 1 & 0 & -2 \end{vmatrix} = (-\sqrt{3})(-2\sqrt{3}) > 0: \underline{\text{MAX}} ; \quad |\overline{H}(-\tfrac{\sqrt{3}}{2}; \tfrac{1}{2}; 1)| = \begin{vmatrix} 0 & -\sqrt{3} & 1 \\ -\sqrt{3} & 0 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 6 > 0: \text{MAX}.$$