# TASK MATHEMATICS for ECONOMIC APPLICATIONS 04/07/2015

I M 1) The complex number $i-1$ has argument $\alpha = 3\pi/4$ and modulus $\rho = \sqrt{2}$ thus $\sqrt[3]{i-1} = \sqrt[3]{\sqrt{2}\left(\cos\frac{3\pi}{4} + i\ \text{sen}\,\frac{3\pi}{4}\right)} =$

$= \sqrt[6]{2}\left(\cos\left(\frac{3\pi}{4}\cdot\frac{1}{3} + \frac{2k\pi}{3}\right) + i\ \text{sen}\left(\frac{3\pi}{4}\cdot\frac{1}{3} + \frac{2k\pi}{3}\right)\right) =$

$= \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2}{3}k\pi\right) + i\ \text{sen}\left(\frac{\pi}{4} + \frac{2}{3}k\pi\right)\right)$ with $k = 0, 1, 2$. The three roots are:

$z_0 = \sqrt[6]{2}\left(\cos\frac{\pi}{4} + i\ \text{sen}\,\frac{\pi}{4}\right) = \frac{\sqrt[3]{4}}{2}(1+i)$;

$z_1 = \sqrt[6]{2}\left(\cos\frac{11}{12}\pi + i\ \text{sen}\,\frac{11}{12}\pi\right) = \sqrt[6]{2}\left(-\frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right) + \frac{\sqrt{2}}{4}i\left(\sqrt{3}-1\right)\right) =$

$= \frac{\sqrt[3]{4}}{4}\left(-\left(\sqrt{3}+1\right) + i\left(\sqrt{3}-1\right)\right)$;

$z_2 = \sqrt[6]{2}\left(\cos\frac{19}{12}\pi + i\ \text{sen}\,\frac{19}{12}\pi\right) = \sqrt[6]{2}\left(\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right) - i\,\frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right)\right) =$

$= \frac{\sqrt[3]{4}}{4}\left(\left(\sqrt{3}-1\right) - i\left(\sqrt{3}+1\right)\right)$.

I M 2) The characteristic polynomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & -\lambda & 1 \\ 4 & -4 & k-\lambda \end{vmatrix} =$

$= (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -4 & k-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 4 & k-\lambda \end{vmatrix} - \begin{vmatrix} 2 & -\lambda \\ 4 & -4 \end{vmatrix} =$

$= (1-\lambda)\left(\lambda^2 - k\lambda + 4\right) - 2(2k - 2\lambda - 4) - (-8 + 4\lambda) =$

$= -\lambda^3 + (1+k)\lambda^2 - (4+k)\lambda + 4(5-k)$;

if $\lambda = 0$ is an eigenvalue of matrix $\mathbb{A}$, $p_{\mathbb{A}}(\lambda)$ must satisfy condition $p_{\mathbb{A}}(0) = 0$ or $4(5-k) = 0$ and so $k = 5$. For $k = 5$, $p_{\mathbb{A}}(\lambda)$ becomes:

$p_{\mathbb{A}}(\lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda = -\lambda(\lambda-3)^2$ and thus the three eigenvalues of $\mathbb{A}$ are $\lambda_1 = 0, \lambda_2 = \lambda_3 = 3$. To check for diagonalizability of the matrix we must find the geometric multiplicity of multiple eigenvalue $\lambda_3 = 3$; to do so, we consider the system:

$\|\mathbb{A} - 3\mathbb{I}\| \cdot v = 0 \Rightarrow \begin{cases} -2v_1 + 2v_2 - v_3 = 0 \\ 2v_1 - 3v_2 + v_3 = 0 \\ 4v_1 - 4v_2 + 2v_3 = 0 \end{cases} \Rightarrow \begin{cases} -2v_1 + 2v_2 - v_3 = 0 \\ 2v_1 - 3v_2 + v_3 = 0 \end{cases}$

(the third equation is a multipler of the first), with solution:

$v = (v_1, 0, -2v_1) = v_1\,(1, 0, -2)$.

As we can simply note, the eigenvectors associated with eigenvalue $\lambda_3 = 3$ belong to a line, thus geometric multiplicity of eigenvalue $\lambda_3 = 3$ is 1, less than its algebraic multiplicity, so the matrix $\mathbb{A}$ is not a digonalizable one.

A basis for the eigenspace associated to $\lambda_3 = 3$ is $\mathcal{B}_{ES(3)} = \{(1, 0, -2)\}$.

I M 3) For a linear map $f\colon \mathbb{R}^4 \to \mathbb{R}^3$, $f(\mathbb{X}) = \mathbb{A}\cdot\mathbb{X}$, Kernel's dimension is given by $\dim\left(\mathbb{R}^4\right) - \dim(\text{Imm}(f)) = 4 - \text{Rank}(\mathbb{A})$, so Kernel's dimension is maximum when $\text{Rank}(\mathbb{A})$ is minimum. To calculate the rank of the matrix we reduce it by elementary operations on its lines:

$\begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & m & 1 \\ 1 & 1 & 1 & k \end{Vmatrix} \overset{\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1}}{\Rightarrow} \begin{Vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & m-1 & 0 \\ 0 & 0 & 0 & k-1 \end{Vmatrix}$ and from the last matrix we get:

$$\text{Rank}(\mathbb{A}) = \begin{cases} 1 & \text{if } m = 1 \text{ and } k = 1 \\ 2 & \text{if } (m = 1 \text{ and } k \neq 1) \text{ or } (m \neq 1 \text{ and } k = 1) \\ 3 & \text{if } m \neq 1 \text{ and } k \neq 1 \end{cases}.$$

Kernel's dimension is maximum when $m = k = 1$ and in this case $\dim(\text{Ker}(f)) = 3$.

The matrix becomes $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{Vmatrix}$.

To find bases for Kernel and Image of the linear map we begin determining the elements of Kernel: $\mathbb{A} \cdot \mathbb{X} = \mathbb{O} \Rightarrow x_1 + x_2 + x_3 + x_4 = 0 \Rightarrow x_4 = -x_1 - x_2 - x_3$.
The elements of Kernel have the form: $\mathbb{X} = (x_1, x_2, x_3, -x_1 - x_2 - x_3) =$
$= x_1 \cdot (1, 0, 0, -1) + x_2 \cdot (0, 1, 0, -1) + x_3 \cdot (0, 0, 1, -1)$; so a basis may be:
$\mathcal{B}_{\text{Ker}(f)} = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1)\}$.
The elements of the Image have the form:
$\mathbb{Y} = \mathbb{A} \cdot \mathbb{X} = (x_1 + x_2 + x_3 + x_4, x_1 + x_2 + x_3 + x_4, x_1 + x_2 + x_3 + x_4) =$
$= (y, y, y) = y \cdot (1, 1, 1)$; so $\mathcal{B}_{\text{Imm}(f)} = \{(1, 1, 1)\}$.

I M 4) Remember that square matrices $\mathbb{A}$ and $\mathbb{B}$ of the same order are similar if a non singular matrix $\mathbb{P}$ exists such that $\mathbb{A} \cdot \mathbb{P} = \mathbb{P} \cdot \mathbb{B}$, or $\mathbb{B} = \mathbb{P}^{-1} \cdot \mathbb{A} \cdot \mathbb{P}$. So:

$$\mathbb{B} = \begin{Vmatrix} 2 & 0 \\ 1 & 2 \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} 1 & 2 \\ 2 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} 2 & 0 \\ 1 & 2 \end{Vmatrix} = \frac{1}{4} \cdot \begin{Vmatrix} 2 & 0 \\ -1 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} 4 & 4 \\ 3 & -2 \end{Vmatrix} \Rightarrow$$
$$\Rightarrow \mathbb{B} = \begin{Vmatrix} 2 & 2 \\ 1/2 & -2 \end{Vmatrix}.$$

II M 1) Since $f(P) = 1$, the condition is satisfied; then $\nabla f(x, y) = (2x, 2y)$, $\nabla f(P) = \left(\sqrt{2}, -\sqrt{2}\right)$ and $\mathbb{H}(f) = \mathbb{H}(f(P)) = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix}$.

Since $f'_y(P) \neq 0$ an implicit function $x \rightarrow y(x)$ exists, $x_0 = \dfrac{1}{\sqrt{2}}$.

We calculate $y'(x_0)$ and $y''(x_0)$ and we get: $y'(x_0) = -\dfrac{f'_x(P)}{f'_y(P)} = 1$ and

$$y''(x_0) = -\frac{f''_{xx}(P) + 2 f''_{xy}(P) \cdot y'(x_0) + f''_{yy}(P) \cdot [y'(x_0)]^2}{f'_y(P)} = -\frac{2 + 0 + 2}{-\sqrt{2}} = 2\sqrt{2}.$$

II M 2) From $\mathcal{D}_v f(1, 1) = \nabla f(1, 1) \cdot v$ we have $\nabla f = (e^{x-y} + xe^{x-y}, -xe^{x-y})$, $\nabla f(1, 1) = (2, -1)$ and $\mathcal{D}_v f(1, 1) = (2, -1) \cdot (\cos\alpha, \text{sen}\,\alpha) = 2\cos\alpha - \text{sen}\,\alpha$.
From condition $\mathcal{D}_v f(1, 1) = 0$ it follows:
$2\cos\alpha - \text{sen}\,\alpha = 0 \Rightarrow \text{tg}\,\alpha = 2 \Rightarrow \alpha = \text{arctg}\,2, \ -\dfrac{\pi}{2} < \alpha < \dfrac{\pi}{2}$.

II M 3) We construct the Lagrangian function:
$\Lambda(x, y, z, \lambda) = x - y + z - \lambda\left(x^2 + y^2 + z^2 - 3\right)$.
First Order Conditions:
$\nabla\Lambda = \left(1 - 2\lambda x, -1 - 2\lambda y, 1 - 2\lambda z, -\left(x^2 + y^2 + z^2 - 3\right)\right)$, from which:

$$\begin{cases} 1-2\lambda x=0 \\ -1-2\lambda y=0 \\ 1-2\lambda z=0 \\ x^2+y^2+z^2=3 \end{cases} \Rightarrow \begin{cases} x=1/(2\lambda) \\ y=-1/(2\lambda) \\ z=1/(2\lambda) \\ 3/(4\lambda^2)=3 \end{cases} \Rightarrow \begin{cases} x=\pm 1 \\ y=\mp 1 \\ z=\pm 1 \\ \lambda=\pm 1/2 \end{cases}.$$

We get two constrained critical points: $P_1=(1,-1,1)$ and $P_2=(-1,1,-1)$.

Second Order Conditions:

$$\overline{\mathbb{H}}(\Lambda)=\left\|\begin{matrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{matrix}\right\|.$$

The problem has three variables and one constraint, so we must consider two leading minors, $\overline{\mathbb{H}}_3$ and $\overline{\mathbb{H}}_4$:

$$\overline{\mathbb{H}}_3=\begin{vmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{vmatrix}=-2x\begin{vmatrix} 2x & 0 \\ 2y & -2\lambda \end{vmatrix}+2y\begin{vmatrix} 2x & -2\lambda \\ 2y & 0 \end{vmatrix}=$$

$$=8\lambda x^2+8\lambda y^2=8\lambda\left(x^2+y^2\right);$$

$$\overline{\mathbb{H}}_4=\begin{vmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{vmatrix}=$$

$$=-2z\begin{vmatrix} 2x & 2y & 2z \\ -2\lambda & 0 & 0 \\ 0 & -2\lambda & 0 \end{vmatrix}-2\lambda\begin{vmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{vmatrix}=$$

$$=-4z^2\left(4\lambda^2\right)-2\lambda[-2x(-4\lambda x)+2y(4\lambda y)]=$$

$$=-16\lambda^2x^2-16\lambda^2y^2-16\lambda^2z^2=-16\lambda^2\left(x^2+y^2+z^2\right).$$

It's easy to note that $\overline{\mathbb{H}}_4<0$ at $P_1$ and $P_2$, $\overline{\mathbb{H}}_3>0$ at $P_1$ while $\overline{\mathbb{H}}_3<0$ at $P_2$.

It follows that $P_1$ is a maximum point with $f(P_1)=3$ while $P_2$ is a minimum point with $f(P_2)=-3$.

II M 4) The feasible region $\mathcal{E}$ is a square represented in red colour in the left figure in the next page. Note that the function $f$ is simmetric respect to the origin $O=(0,0)$ $(f(-x,-y)=f(x,y)$ and $f(-x,y)=f(x,-y))$ thus we can study the optimization problem only on the right part of the square $\mathcal{E}$.

First case (free optimization)

First Order Conditions: $\nabla f=(2x-4y,-4x+2y)$. $\begin{cases} 2x-4y=0 \\ -4x+2y=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$.

Free optimization presents only one critical point $O$ that belongs to the interior of $\mathcal{E}$.

Second Order Conditions: $|\mathbb{H}(f)|=|\mathbb{H}_2(O)|=\begin{vmatrix} 2 & -4 \\ -4 & 2 \end{vmatrix}=-12<0$; so $O$ is a saddle point.

Second case (optimization along the border of $\mathcal{E}$)

$f(x,1-x)=x^2-4x(1-x)+(1-x)^2=6x^2-6x+1$, that is a parabola with minimum equal $-1/2$ at point $x^*=1/2$.

$f(x,x-1)=x^2-4x(x-1)+(x-1)^2=-2x^2+2x+1$, that is a parabola with maximum equal $3/2$ at point $x^*=1/2$.

We conclude that $\text{Max}(f) = 3/2$ at points $(1/2, -1/2)$ and $(-1/2, 1/2)$ (blue points on the left figure below), $\min(f) = -1/2$ at points $(1/2, 1/2)$ and $(-1/2, -1/2)$ (pink points).
On the right figure below there have been drawn negative level curves (pink), zero level curves (yellow) and positive level curves (blue).