TASK MATHEMATICS FOR ECONOMIC APPLICATIONS 22/9/2015 $\boxed{\text{MfEA1}}$

IM1) $x^3-5x^2+4x-20=0$. For $x=5$: $125-125+20-20=0$.

$$\begin{array}{c|ccc|c} & 1 & -5 & 4 & -20 \\ 5 & & 5 & 0 & 20 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

$x^3-5x^2+4x-20=(x-5)(x^2+4)=0 \Rightarrow x_1=5;\ x_2=2i;\ x_3=-2i.$

$x_1=5 \Rightarrow x_1=5(\cos 0+i\,\mathrm{sen}\,0)$. $\sqrt[3]{5}=\sqrt[3]{5}\left(\cos k\cdot\frac{2\pi}{3}+i\,\mathrm{sen}\,k\cdot\frac{2\pi}{3}\right);\ 0\le k\le 2.$

$\underline{\text{For } k=0}$: $\sqrt[3]{5}(\cos 0+i\,\mathrm{sen}\,0)=\sqrt[3]{5}$; $\underline{\text{For } k=1}$: $\sqrt[3]{5}\left(\cos\frac{2}{3}\pi+i\,\mathrm{sen}\,\frac{2}{3}\pi\right)=\sqrt[3]{5}\left(-\frac{1}{2}+i\cdot\frac{\sqrt{3}}{2}\right)$;

$\underline{\text{For } k=2}$: $\sqrt[3]{5}\left(\cos\frac{4}{3}\pi+i\,\mathrm{sen}\,\frac{4}{3}\pi\right)=\sqrt[3]{5}\cdot\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right).$

IM2) $\left\|\begin{matrix}2&2&1\\0&1&m\\1&2&k\end{matrix}\right\| \to \begin{vmatrix}2-\lambda&2&1\\0&1-\lambda&m\\1&2&k-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda&0&1-k+\lambda\\0&1-\lambda&m\\1&2&k-\lambda\end{vmatrix}=(1-\lambda)\left[(1-\lambda)(k-\lambda)-2m\right]+1\left(0-(1-\lambda)(1-k+\lambda)\right)=$

$=(1-\lambda)(\lambda^2-k\lambda-\lambda+k-2m-1+k-\lambda)=(1-\lambda)(\lambda^2-(k+2)\lambda+2k-2m-1)=0\Rightarrow\lambda_1=1.$

For $\lambda=1$: $1-k-2+2k-2m-1=0\Rightarrow k=2m+2\Rightarrow\lambda^2-(2m+4)\lambda+2m+3=0.$

$\Delta=0$: $m^2+4+4m-2m-3=m^2+2m+1=(m+1)^2=0$ for $m=-1\Rightarrow k=0.$

For $k=0$ and $m=-1$: $\lambda_1=\lambda_2=\lambda_3=1$: $m_1^a=3$.

For $k=2m+2$; $m\ne-1$: $\lambda_1=\lambda_2=1$: $m_1^a=2$.

IM3) $\begin{cases}(1;0;1)\cdot(x;y;z)=0\\(0;1;0)\cdot(x;y;z)=0\end{cases}\Rightarrow\begin{cases}x+z=0\\y=0\end{cases}\Rightarrow\begin{cases}z=-x\\y=0\end{cases}\Rightarrow X_3=(1;0;-1).$

Orthonormal basis (unit vectors): $X=\left\{\left(\frac{1}{\sqrt2};0;\frac{1}{\sqrt2}\right);(0;1;0);\left(\frac{1}{\sqrt2};0;-\frac{1}{\sqrt2}\right)\right\}.$

$$\left\|\begin{matrix}\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&1&0\\\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\left\|\begin{matrix}1\\1\\1\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\left\|\begin{matrix}\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&1&0\\\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\end{matrix}\right\|\cdot\left\|\begin{matrix}1\\1\\1\end{matrix}\right\|=\left\|\begin{matrix}\frac{1}{\sqrt2}+\frac{1}{\sqrt2}\\0+1+0\\\frac{1}{\sqrt2}-\frac{1}{\sqrt2}\end{matrix}\right\|=\left\|\begin{matrix}\sqrt2\\1\\0\end{matrix}\right\|\quad\left(\underline{\underline{U^{-1}=U^T}}\right).$$

IM4) $A=\left\|\begin{matrix}0&0&2\\0&0&2\\2&2&0\end{matrix}\right\|\to\begin{vmatrix}-\lambda&0&2\\0&-\lambda&2\\2&2&-\lambda\end{vmatrix}=(-\lambda)(\lambda^2-4)+2(2\lambda)=(-\lambda)(\lambda^2-4-4)=(-\lambda)(\lambda^2-8)=0.$

$\lambda_1=0;\ \lambda_2=2\sqrt2;\ \lambda_3=-2\sqrt2.$

$\underline{\text{For } \lambda=0}$ $\left\|\begin{matrix}0&0&2\\0&0&2\\2&2&0\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\underline{0}\Rightarrow\begin{cases}2z=0\\2z=0\\2x+2y=0\end{cases}\Rightarrow\begin{cases}y=-x\\z=0\end{cases}\Rightarrow X_1=(1;-1;0).$

MfEA2

For $\lambda = 2\sqrt{2}$ $\left\|\begin{matrix} -2\sqrt{2} & 0 & 2 \\ 0 & -2\sqrt{2} & 2 \\ 2 & 2 & -2\sqrt{2}\end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z\end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases} -2\sqrt{2}x + 2z = 0 \\ -2\sqrt{2}y + 2z = 0 \\ 2x + 2y - 2\sqrt{2}z = 0\end{cases} \Rightarrow \begin{cases} y = x \\ z = \sqrt{2}x \end{cases} \Rightarrow X_2 = (1; 1; \sqrt{2})$.

For $\lambda = -2\sqrt{2}$ $\left\|\begin{matrix} 2\sqrt{2} & 0 & 2 \\ 0 & 2\sqrt{2} & 2 \\ 2 & 2 & 2\sqrt{2}\end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z\end{matrix}\right\| = \underline{0} \Rightarrow \begin{cases} 2\sqrt{2}x + 2z = 0 \\ 2\sqrt{2}y + 2z = 0 \\ 2x + 2y + 2\sqrt{2}z = 0\end{cases} \Rightarrow \begin{cases} y = x \\ z = -\sqrt{2}x \end{cases} \Rightarrow X_3 = (1; 1; -\sqrt{2})$.

And so: $P = \left\|\begin{matrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2}\end{matrix}\right\| \Rightarrow A \cdot P = P \cdot D$ with $D = \left\|\begin{matrix} 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 \\ 0 & 0 & -2\sqrt{2}\end{matrix}\right\|$.

II M1) $f(x;y) = y^2 e^x - x^2 y$:

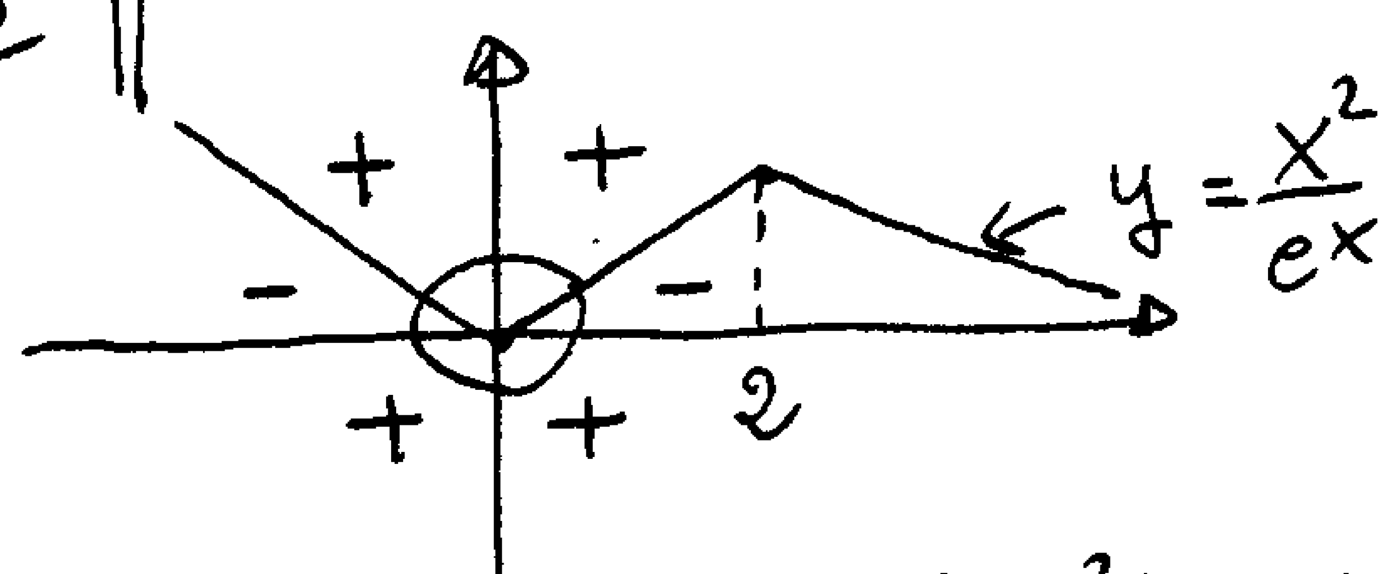
$\begin{cases} f'_x = y^2 e^x - 2xy = y \cdot (y e^x - 2x) = 0 \\ f'_y = 2y e^x - x^2 = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$ or $\begin{cases} \frac{x^2}{2e^x} \cdot e^x - 2x = \frac{x^2 - 4x}{2} = 0 \\ y = \frac{x^2}{2e^x} \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ or $\begin{cases} x = 4 \\ y = \frac{8}{e^4} \end{cases}$. $H(x;y) = \left\|\begin{matrix} y^2 e^x - 2y & 2y e^x - 2x \\ 2y e^x - 2x & 2e^x \end{matrix}\right\|$

$H\left(4; \frac{8}{e^4}\right) = \left\|\begin{matrix} \frac{64}{e^8} \cdot e^4 - \frac{16}{e^4} & \frac{16}{e^4} \cdot e^4 - 8 \\ \frac{16}{e^4} \cdot e^4 - 8 & 2 \cdot e^4 \end{matrix}\right\| = \left\|\begin{matrix} \frac{48}{e^4} & 8 \\ 8 & 2e^4 \end{matrix}\right\| \Rightarrow \begin{cases} \frac{48}{e^4} > 0 ; 2e^4 > 0 \\ \frac{48}{e^4} \cdot 2 \cdot e^4 - 64 = 32 > 0 \end{cases}$ Minimum Point.

$H(0;0) = \left\|\begin{matrix} 0 & 0 \\ 0 & 2 \end{matrix}\right\|$: Not a Maximum P. $f(x;y) = y(y e^x - x^2) \geq 0 \Rightarrow$

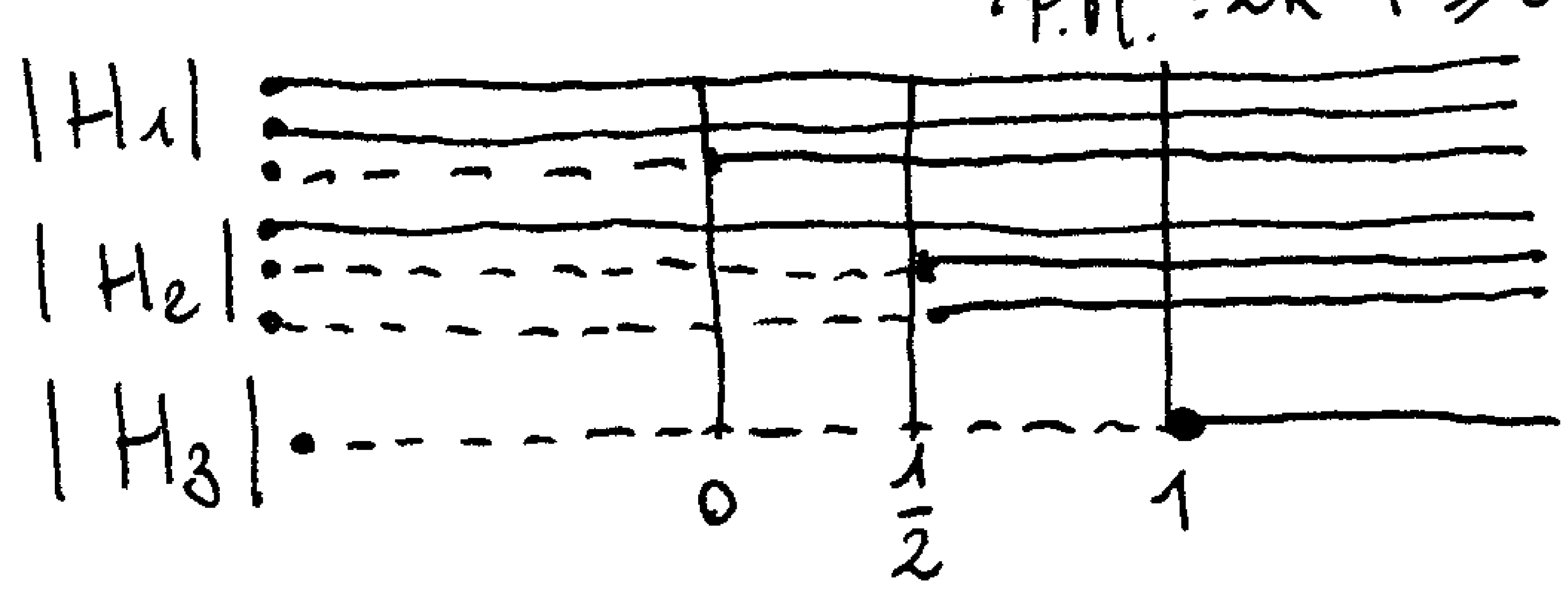
$\Rightarrow \begin{cases} y \geq 0 \\ y \geq \frac{x^2}{e^x} \end{cases}$



(0;0) is a Saddle Point.

II M2) $2(dx)^2 + 2(dy)^2 + k \cdot (dz)^2 - 2dx\,dz + 2dy\,dz = \|dx\,dy\,dz\| \cdot \left\|\begin{matrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & k \end{matrix}\right\| \cdot \left\|\begin{matrix} dx \\ dy \\ dz \end{matrix}\right\|$.

$|H_1| : \begin{cases} 2 > 0 \\ 2 > 0 \\ k > 0 \end{cases}$; $|H_2| = \begin{cases} 4 > 0 \\ 2k - 1 > 0 \Rightarrow k > \frac{1}{2} \\ \text{P.M.} : 2k - 1 \geq 0 : k \geq \frac{1}{2} \end{cases}$; $|H_3| = 2(2k-1) - 1(2) = 4k - 4 > 0 \Rightarrow k > 1$



For $k > 1$: Q.f. positive definite.
For $k = 1$: Q.f. positive semi-definite
Otherwise: Q.f. indefinite.

MfEA3

II n3) $\begin{cases} \max/\min f(x;y) = x+y \\ \text{u.c. } xy = k \end{cases}$. $\Lambda = x+y-\lambda(xy-k)$.

$$\begin{cases} \Lambda'_x = 1-\lambda y = 0 \\ \Lambda'_y = 1-\lambda x = 0 \\ xy = k \end{cases} \Rightarrow \begin{cases} y = \frac{1}{\lambda} \\ x = \frac{1}{\lambda} \\ \frac{1}{\lambda^2} = k \end{cases} \Rightarrow \begin{cases} y = \frac{1}{\lambda} \\ x = \frac{1}{\lambda} \\ \lambda = \pm\frac{1}{\sqrt{k}} \end{cases}$$

. We have solutions only if $k>0$.

$P_1 = \left(\sqrt{k}; \sqrt{k}; \frac{1}{\sqrt{k}}\right)$: $f(P_1) = 2\sqrt{k} > 0$: maximum

$P_2 = \left(-\sqrt{k}; -\sqrt{k}; -\frac{1}{\sqrt{k}}\right)$: $f(P_2) = -2\sqrt{k} < 0$: Minimum.

II n4) $\begin{cases} f(x;y;z) = e^{x^2-2y^2-1} - z = 0 \\ g(x;y;z) = z^3 + y^4 - x^2 = 0 \end{cases}$. $\begin{cases} f(1;0;1) = 0 \\ g(1;0;1) = 0 \end{cases}$.

$$\frac{\partial(f;g)}{\partial(x;y;z)} = \left\| \begin{matrix} 2x e^{x^2-2y^2-1} & -4y e^{x^2-2y^2-1} & -1 \\ -2x & 4y^3 & 3z^2 \end{matrix} \right\|$$

$$\frac{\partial(f;g)}{\partial(x;y;z)}(1;0;1) = \left\| \begin{matrix} 2 & 0 & -1 \\ -2 & 0 & 3 \end{matrix} \right\|$$

Since $\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} = 6-2 = 4 \neq 0$ it is possible to define $F: y \to (x(y); z(y))$.

$$\frac{dx}{dy} = -\frac{\begin{vmatrix} 0 & -1 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix}} = 0 ; \quad \frac{dz}{dy} = -\frac{\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix}} = 0.$$

So at point $y=0$ there is no tangent vector.