Intermediate test for Mathematics for Economic Applications 9/1/2016 $\boxed{\text{MfEA1}}$

IM1) $(i-1)^4 = (1-x)^3 \Rightarrow 1-x = \sqrt[3]{(i-1)^4}$.

$i-1 = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\cdot\left(\cos\frac{3}{4}\pi + i \operatorname{sen}\frac{3}{4}\pi\right)$.

$(i-1)^4 = (\sqrt{2})^4\cdot\left(\cos 4\cdot\frac{3}{4}\pi + i \operatorname{sen} 4\cdot\frac{3}{4}\pi\right) = 4\cdot\left(\cos 3\pi + i\operatorname{sen} 3\pi\right) = 4(-1) = -4$.

$1-x = \sqrt[3]{-4}$. $-4 = 4\cdot(\cos\pi + i\operatorname{sen}\pi)$. $x = 1-\sqrt[3]{-4}$.

$\sqrt[3]{-4} = \sqrt[3]{4}\cdot\left(\cos\left(\frac{\pi}{3} + k\cdot\frac{2\pi}{3}\right) + i\operatorname{sen}\left(\frac{\pi}{3}+k\cdot\frac{2\pi}{3}\right)\right);\ 0\le k\le 2$.

$\underline{k=0}$: $\sqrt[3]{4}\cdot\left(\cos\frac{\pi}{3} + i\operatorname{sen}\frac{\pi}{3}\right) = \sqrt[3]{4}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \Rightarrow x_0 = 1-\sqrt[3]{4}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$.

$\underline{k=1}$: $\sqrt[3]{4}\cdot\left(\cos\pi + i\operatorname{sen}\pi\right) = -\sqrt[3]{4} \Rightarrow x_1 = 1+\sqrt[3]{4}$.

$\underline{k=2}$: $\sqrt[3]{4}\cdot\left(\cos\frac{5}{3}\pi + i\operatorname{sen}\frac{5}{3}\pi\right) = \sqrt[3]{4}\cdot\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) \Rightarrow x_2 = 1-\sqrt[3]{4}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$.

IM2) $A = \begin{Vmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & k \end{Vmatrix} \to \begin{vmatrix} 3-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & k-\lambda \end{vmatrix} = (2-\lambda)\cdot\left((3-\lambda)(k-\lambda)-1\right) = (2-\lambda)\left(\lambda^2-(3+k)\lambda+3k-1\right) = 0$.

$\lambda = 2$ is eigenvalue $\forall k$. $\lambda^2-(3+k)\lambda+3k-1 = 0$ for $\lambda=2$: $4-6-2k+3k-1 = k-3 = 0 \Rightarrow$

$\Rightarrow k=3$. For $k=3$: $\lambda^2-6\lambda+8 = (\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda_1=\lambda_2 = 2$; $\lambda_3 = 4$.

For $k=3$ and $\lambda=2$: $\|A-2\mathbb{I}\| = \begin{Vmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{Vmatrix} \Rightarrow \text{RANK}(A-2\mathbb{I}) = 1 \Rightarrow m_2^g = 3-1 = 2 = m_2^a \Rightarrow$

$\Rightarrow$ For $k=3$ the matrix A is a diagonalizable one.

•) $\lambda^2-(3+k)\lambda+3k-1 = 0 \Rightarrow \lambda = \dfrac{(3+k)\pm\sqrt{9+k^2+6k-12k+4}}{2} \Rightarrow \Delta = k^2-6k+13$.

$\Delta = 0$: $k = 3\pm\sqrt{9-13}$. So $\Delta > 0\ \forall k\in\mathbb{R}$. The unique multiple eigenvalue is $\lambda = 2$; the matrix is diagonalizable $\forall k\in\mathbb{R}$ and the matrix cannot have complex eigenvalues.

IM3) $\begin{Vmatrix} k_1 & 1 & 1 \\ -1 & k_2 & 2 \\ 1 & -1 & k_3 \end{Vmatrix}$. To get an orthogonal matrix we need orthogonal unit vectors.

$\begin{cases} (k_1;-1;1)\cdot(1;k_2;-1) = 0 \\ (k_1;-1;1)\cdot(1;2;k_3) = 0 \\ (1;k_2;-1)\cdot(1;2;k_3) = 0 \end{cases} \Rightarrow \begin{cases} k_1-k_2-1 = 0 \\ k_1-2+k_3 = 0 \\ 1+2k_2-k_3 = 0 \end{cases} \Rightarrow \begin{cases} k_1-k_2 = 1 \\ k_1+k_3 = 2 \\ 2k_2-k_3 = -1 \end{cases}$. $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} = -3 \ne 0$.

From Cramer's Theorem:

$$K_1 = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 2 & -1 \end{vmatrix}}{-3} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ -1 & 1 & -1 \end{vmatrix}}{-3} = \frac{1\cdot(-2-1)}{-3} = 1; \quad X_1 = (1;-1;1); \ \|X_1\| = \sqrt{3};$$

$$K_2 = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{vmatrix}}{-3} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{vmatrix}}{-3} = \frac{0}{-3} = 0; \quad X_2 = (1;0;-1); \ \|X_2\| = \sqrt{2};$$

$$K_3 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & -1 \end{vmatrix}}{-3} = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix}}{-3} = \frac{1\cdot(-1-2)}{-3} = \frac{-3}{-3} = 1. \quad X_3 = (1;2;1); \ \|X_3\| = \sqrt{6}.$$

The orthogonal matrix may be: $U = \begin{Vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{Vmatrix}$.

IM4) $f: \mathbb{R}^3 \to \mathbb{R}^4$; $f(X) = A\cdot X$.

$$A = \begin{Vmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ 1 & -2 & K \\ K & 1 & 1 \end{Vmatrix} \xrightarrow[\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - K\cdot R_1}]{} \begin{Vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & K-1 \\ 0 & 1+K & 1-K \end{Vmatrix} \xrightarrow[\substack{R_3 \leftarrow R_3 + R_2 \\ R_4 \leftarrow R_4 + (1+K)R_3}]{} \begin{Vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & K \\ 0 & 0 & K^2-K \end{Vmatrix}.$$

$K^2 - K = K(K-1) = 0 \Rightarrow K = 0$ and $K = 1$.

For $K=1$: $\begin{Vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix}$: Rank(A) = 3, Dim(Ker) = 3−3 = 0; For $K=0$: $\begin{Vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$: Rank(A) = 2, Dim(Ker) = 3−2 = 1: MAX.

To find a basis for the kernel:

$$\begin{Vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_2 \\ x_3 = -x_2 \end{cases} \Rightarrow (2x_2; x_2; -x_2).$$

A basis for the kernel is the vector $X = (2;1;-1)$.

MfEA3

To find a basis for the Image:

$$\left\|\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 2 & -1 & 3 & y_2 \\ 1 & -2 & 0 & y_3 \\ 0 & 1 & 1 & y_4 \end{array}\right\| \to \left\|\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 1 & 1 & y_2-2y_1 \\ 0 & -1 & -1 & y_3-y_1 \\ 0 & 1 & 1 & y_4 \end{array}\right\| \to \left\|\begin{array}{ccc|c} 1 & -1 & 1 & y_1 \\ 0 & 1 & 1 & y_2-2y_1 \\ 0 & 0 & 0 & y_2-3y_1+y_3 \\ 0 & 0 & 0 & y_4+y_3-y_1 \end{array}\right\| \Rightarrow$$

$$\begin{cases} y_3+y_2-3y_1=0 \\ y_4+y_3-y_1=0 \end{cases} \Rightarrow \begin{cases} y_3=3y_1-y_2 \\ y_4=y_1-y_3=y_2-2y_1 \end{cases} : Y=(y_1;y_2;3y_1-y_2;y_2-2y_1).$$

A basis for the Image is $Y=\{(1;0;3;-2);(0;1;-1;1)\}$.

IM5) A similar to B by P: $A\cdot P=P\cdot B$.

$$\left\|\begin{matrix} 1 & 2 \\ 3 & -1 \end{matrix}\right\| \cdot \left\|\begin{matrix} m & 1 \\ 2 & k \end{matrix}\right\| = \left\|\begin{matrix} m & 1 \\ 2 & k \end{matrix}\right\| \cdot \left\|\begin{matrix} 0 & 1 \\ 7 & 0 \end{matrix}\right\| \Rightarrow \begin{cases} m+4=7 \\ 1+2k=m \\ 3m-2=7k \\ 3-k=2 \end{cases} \Rightarrow \begin{cases} m=3 \\ 1+2=3 \\ 9-2=7 \\ k=1 \end{cases}. \quad P=\left\|\begin{matrix} 3 & 1 \\ 2 & 1 \end{matrix}\right\|.$$