Task Mathematics for Economic Applications 27/1/2016 $\boxed{\text{MfEA1}}$

IM1) $1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\cdot\frac{1}{\sqrt{2}}\right)=\sqrt{2}\cdot\left(\cos\frac{\pi}{4}+i\,\text{sen}\,\frac{\pi}{4}\right)$.

$\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)=2\cdot\left(\cos\frac{\pi}{6}+i\,\text{sen}\,\frac{\pi}{6}\right)$.

$$\frac{1+i}{\sqrt{3}+i}=\frac{\sqrt{2}}{2}\cdot\frac{\cos\frac{\pi}{4}+i\,\text{sen}\,\frac{\pi}{4}}{\cos\frac{\pi}{6}+i\,\text{sen}\,\frac{\pi}{6}}=\frac{1}{\sqrt{2}}\cdot\left(\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)+i\,\text{sen}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)\right)=\frac{1}{\sqrt{2}}\cdot\left(\cos\frac{\pi}{12}+i\,\text{sen}\,\frac{\pi}{12}\right).$$

$$\frac{1+i}{\sqrt{3}+i}=\frac{(1+i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)}=\frac{(\sqrt{3}+1)+i(\sqrt{3}-1)}{4}=\frac{1}{\sqrt{2}}\cdot\left(\frac{\sqrt{3}+1}{2\sqrt{2}}+i\,\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\Rightarrow\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}\,;\ \text{sen}\,\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

IM2) If Y is a linear combination of X_1 and X_2, the determinant of the matrix having X_1, X_2 and Y as columns must be equal to 0:

$$\begin{vmatrix}1&-2&0\\-1&m&1\\2&2&k\end{vmatrix}=1\cdot\begin{vmatrix}m&1\\2&k\end{vmatrix}+2\begin{vmatrix}-1&1\\2&k\end{vmatrix}=mk-2-2k-4=mk-2k-6=0\Rightarrow$$

$\Rightarrow m=\frac{2k+6}{k}$ or $k=\frac{6}{m-2}$ is the relation.

IM3) $$\begin{Vmatrix}2&0&1\\2&1&2\\1&-3&-4\end{Vmatrix}\to\begin{vmatrix}2-\lambda&0&1\\2&1-\lambda&2\\1&-3&-4-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda&0&1\\0&1-\lambda&2\\5+\lambda&-3&-4-\lambda\end{vmatrix}=(1-\lambda)\big((1-\lambda)(-4-\lambda)+6\big)+(5+\lambda)\cdot(\lambda-1)=$$

$$=(1-\lambda)(\lambda^2+3\lambda+2-5-\lambda)=(1-\lambda)(\lambda^2+2\lambda-3)=(1-\lambda)(\lambda-1)(\lambda+3)=0\Rightarrow\lambda_1=\lambda_2=1;\ \lambda_3=-3.$$

$$\|A-1\cdot\mathbb{I}\|=\begin{Vmatrix}1&0&1\\2&0&2\\1&-3&-5\end{Vmatrix}.\ \text{RANK}(A-1\cdot\mathbb{I})=2\Rightarrow m_1^g=3-2=1<2=m_1^a.$$

The matrix is not diagonalizable.

IM4) The linear map is given by the matrix $A=\begin{Vmatrix}1&1&1&1&1\\1&1&1&1&-1\\1&1&1&-1&1\end{Vmatrix}$.

$$\begin{Vmatrix}1&1&1&1&1\\1&1&1&1&-1\\1&1&1&-1&1\end{Vmatrix}\to\begin{Vmatrix}1&1&1&1&1\\0&0&0&0&-2\\0&0&0&-2&0\end{Vmatrix}\Rightarrow\text{RANK}(A)=3\Rightarrow\begin{cases}\text{Dim}(\text{Imm}(A))=3\\\text{Dim}(\text{Ker}(A))=5-3=2.\end{cases}$$

A basis for the Kernel:

$$\begin{Vmatrix}1&1&1&1&1\\0&0&0&0&-2\\0&0&0&-2&0\end{Vmatrix}\cdot\begin{Vmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{Vmatrix}=\mathbb{O}\Rightarrow\begin{cases}x_1+x_2+x_3+x_4+x_5=0\\-2x_5=0\\-2x_4=0\end{cases}\Rightarrow\begin{cases}x_3=-x_1-x_2\\x_4=0\\x_5=0\end{cases}.$$

MfEA 2

$X \in \mathrm{Ker}(A)$ if $X = (x_1; x_2; -x_1 - x_2; 0; 0)$

A basis for Ker(A) is: $\{(1;0;-1;0;0); (0;1;-1;0;0)\}$.

A basis for the Image: Since $f: \mathbb{R}^5 \to \mathbb{R}^3$ and Dim (Imm(A)) = 3, the map is surjective, and so every basis for $\mathbb{R}^3$ is a basis for Imm(A).

II M1) $f(x;y) = \log(x^2+y^2) - \frac{y}{x} = 0$. $f(1;0) = 0$.

$$\nabla f = \left(\frac{2x}{x^2+y^2} + \frac{y}{x^2}; \frac{2y}{x^2+y^2} - \frac{1}{x}\right). \quad \nabla f(1;0) = (2+0; 0-1) = (2;-1).$$

So $y'(1) = -\frac{f'_x}{f'_y} = -\frac{2}{-1} = 2$.

II M2) $f(x;y) = e^{x-y}$: Continuous and twice differentiable function.

$\nabla f(x;y) = (e^{x-y}; -e^{x-y})$; $\nabla f(0;0) = (1;-1)$. $\mathcal{D}_v f(0;0) = (1;-1)\cdot(\cos\alpha; \operatorname{sen}\alpha) = \cos\alpha - \operatorname{sen}\alpha$.

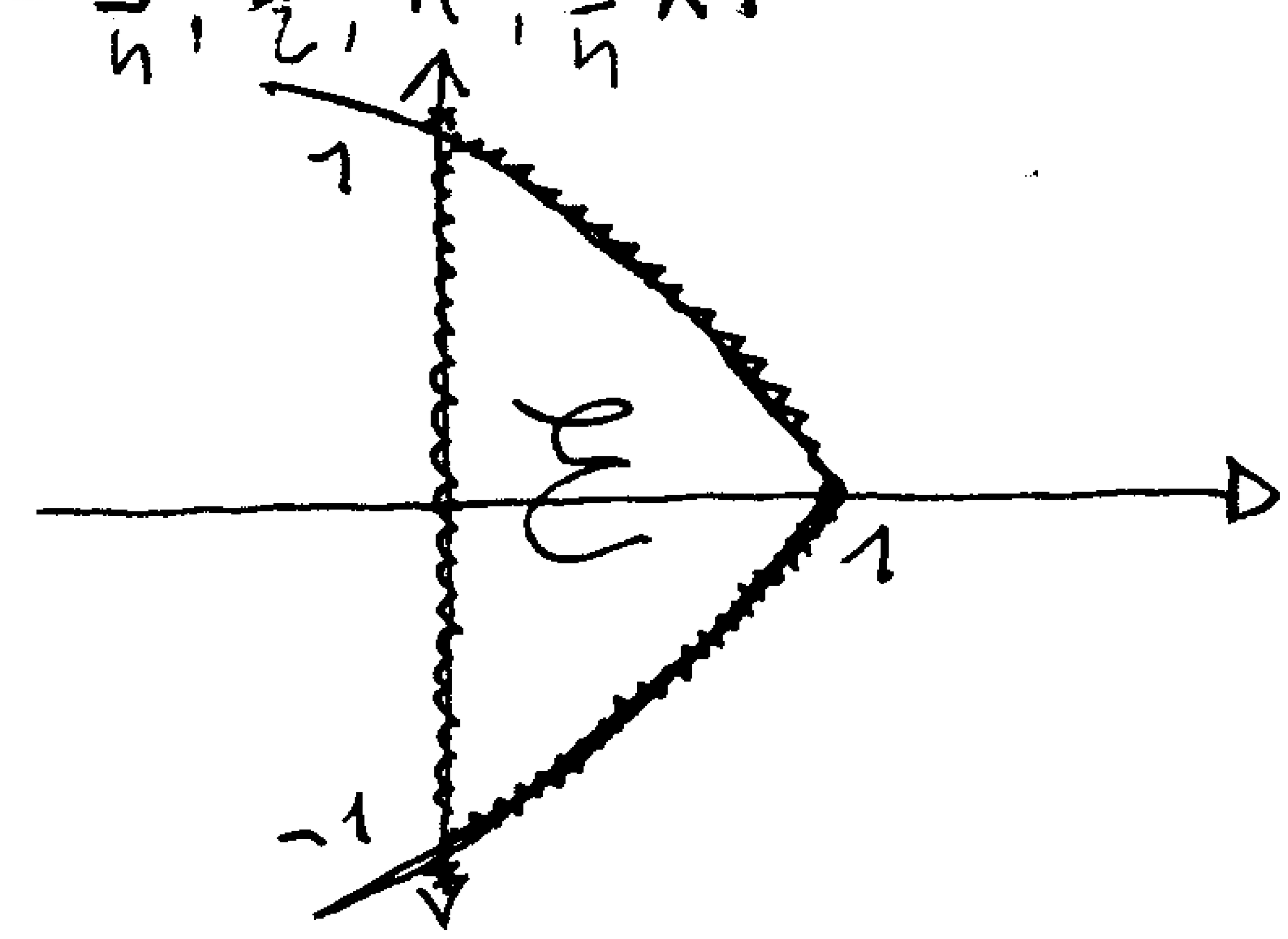
$$H(f(x;y)) = \begin{Vmatrix} e^{x-y} & -e^{x-y} \\ -e^{x-y} & e^{x-y} \end{Vmatrix}; \quad H(f(0;0)) = \begin{Vmatrix} 1 & -1 \\ -1 & 1 \end{Vmatrix}. \quad \mathcal{D}^2_{v,v} f(0;0) = \begin{Vmatrix} \cos\alpha & \operatorname{sen}\alpha \end{Vmatrix} \cdot \begin{Vmatrix} 1 & -1 \\ -1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} -\cos\alpha \\ -\operatorname{sen}\alpha \end{Vmatrix} =$$

$$= \begin{Vmatrix} \cos\alpha & \operatorname{sen}\alpha \end{Vmatrix} \cdot \begin{Vmatrix} \operatorname{sen}\alpha - \cos\alpha \\ \cos\alpha - \operatorname{sen}\alpha \end{Vmatrix} = \cos\alpha \operatorname{sen}\alpha - \cos^2\alpha + \operatorname{sen}\alpha\cos\alpha - \operatorname{sen}^2\alpha =$$

$$= 2\operatorname{sen}\alpha\cos\alpha - 1 = \operatorname{sen}2\alpha - 1. \quad \mathcal{D}_v f(0;0) = \mathcal{D}^2_{v,v} f(0;0) \Rightarrow \operatorname{sen}2\alpha - 1 = \cos\alpha - \operatorname{sen}\alpha.$$

Equality is satisfied at least for $\alpha = \frac{\pi}{4}; \frac{\pi}{2}; \pi; \frac{5}{4}\pi$.

II M3) $\begin{cases} \text{Max/min } f(x;y) = xy - y^2 \\ \text{u.c.} \begin{cases} x \le 1 - y^2 \\ x \ge 0 \end{cases} \Rightarrow \begin{cases} x + y^2 - 1 \le 0 \\ -x \le 0 \end{cases} \end{cases}$.



$f(x;y)$ is a continuous function, $\mathcal{E}$ is a limited and closed set. Constraints are qualified.

$$\Lambda = xy - y^2 - \lambda_1(x + y^2 - 1) - \lambda_2(-x).$$

MfEA3

For $\lambda_1 = \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = y = 0 \\ \Lambda'_y = x - 2y = 0 \\ x \leq 1 - y^2 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \leq 1 \\ 0 \geq 0 \end{cases} \cdot \quad H = \begin{Vmatrix} 0 & 1 \\ 1 & -2 \end{Vmatrix}; \quad |H| = -1 < 0 : \text{Saddle point.}$$

For $\lambda_1 \neq 0; \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = y - \lambda_1 = 0 \\ \Lambda'_y = x - 2y - 2\lambda_1 y = 0 \\ x = 1 - y^2 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = y \\ x = 2y + 2y^2 \\ 2y + 2y^2 = 1 - y^2 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = y \\ x = 2y + 2y^2 \\ 3y^2 + 2y - 1 = 0 \\ x \geq 0 \end{cases}; \quad 3y^2 + 2y - 1 = 0 \Rightarrow$$

$$\Rightarrow y = \frac{-1 \pm \sqrt{1+3}}{3} = \frac{-1 \pm 2}{3} \begin{cases} y = -1 \\ y = \frac{1}{3} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \\ \lambda_1 = -1 \\ 0 \geq 0 \end{cases} \text{Min ?} \cup \begin{cases} x = \frac{8}{9} \\ y = \frac{1}{3} \\ \lambda_1 = \frac{1}{3} \\ \frac{8}{9} \geq 0 \end{cases} \text{Max ?}$$

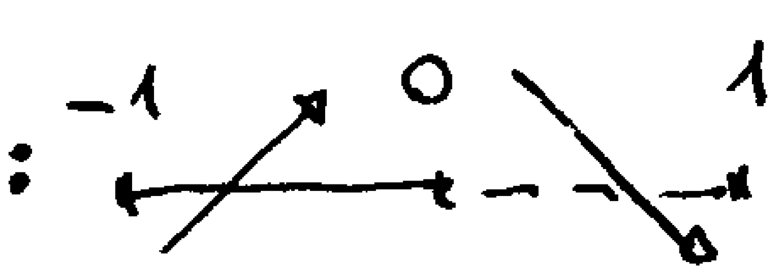
For $\lambda_1 = 0; \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = y + \lambda_2 = 0 \\ \Lambda'_y = x - 2y = 0 \\ x = 0 \\ x \leq 1 - y^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_2 = 0 \\ 0 \leq 1 \end{cases} \quad \text{Just studied.}$$

For $\lambda_1 \neq 0; \lambda_2 \neq 0$

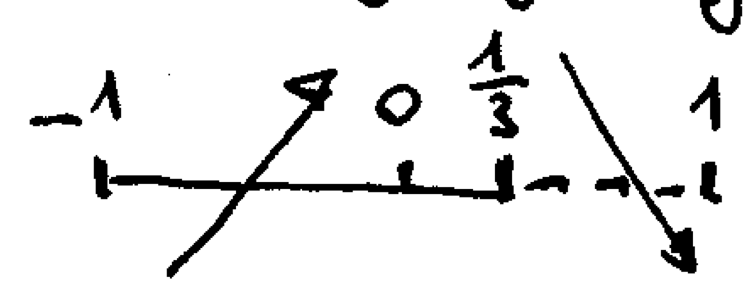
$$\begin{cases} \Lambda'_x = y - \lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = x - 2y - 2\lambda_1 y = 0 \\ \begin{cases} x = 0 \\ x = 1 - y^2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \cup \begin{cases} x = 0 \\ y = -1 \end{cases} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ 1 - \lambda_1 + \lambda_2 = 0 \\ 0 - 2 - 2\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = -1 \\ \lambda_2 = -2 \end{cases} \text{Min ?} \cup \begin{cases} x = 0 \\ y = -1 \\ \lambda_1 = -1 \\ \lambda_2 = 0 \end{cases} \text{Just studied. Min ?}$$

For $x = 0$: $f(y) = -y^2$; $f'(y) = -2y \geq 0 \Rightarrow y \leq 0$:



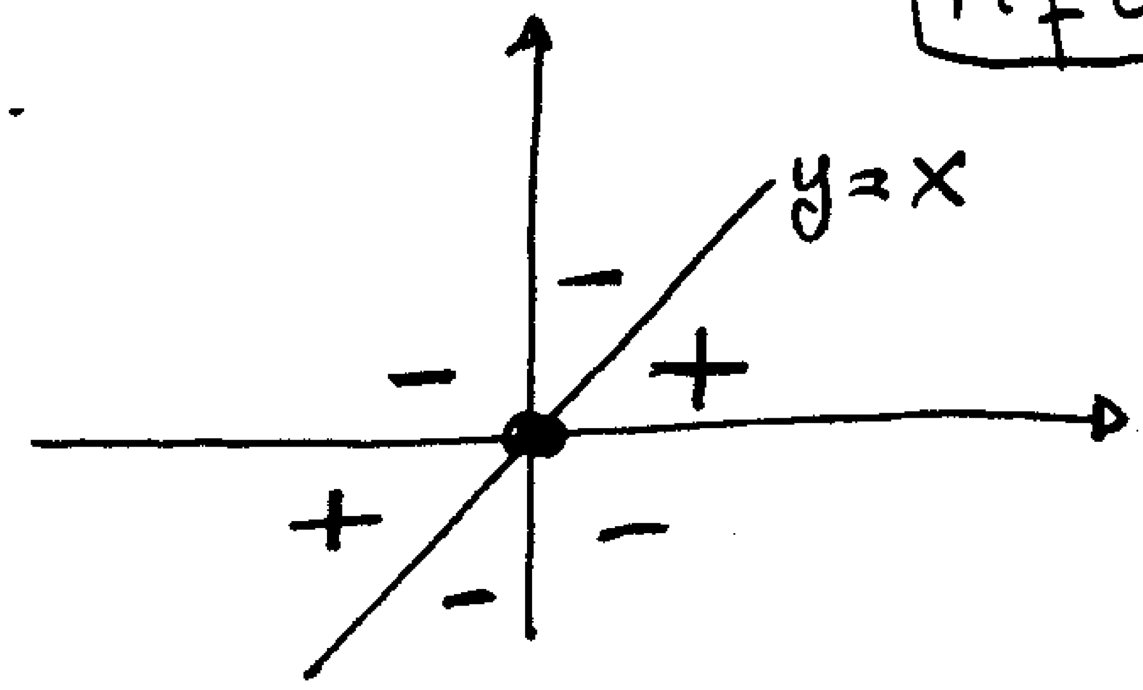
For $x = 1 - y^2$: $f(y) = (1 - y^2) \cdot y - y^2 = y - y^3 - y^2$. $f'(y) = 1 - 3y^2 - 2y \geq 0 \Rightarrow 3y^2 + 2y - 1 \leq 0 \Rightarrow$

$\Rightarrow -1 \leq y \leq \frac{1}{3}$:



MfEA4

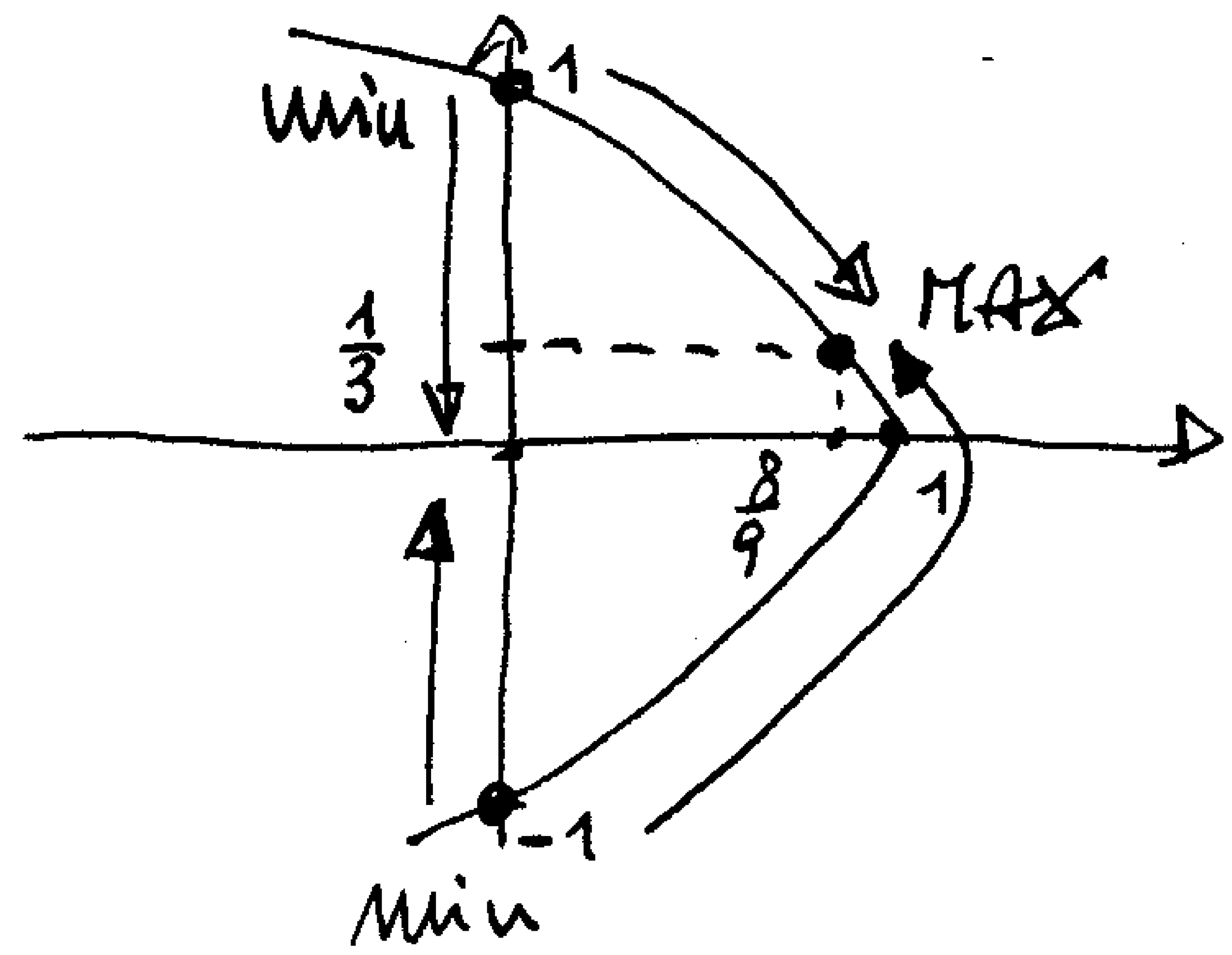
$f(x;y) = y\cdot(x-y) \geq 0 \Rightarrow \begin{cases} y \geq 0 \\ x \geq y \end{cases} \cup \begin{cases} y \leq 0 \\ x \leq y \end{cases}$.



$f(0;0) = 0 \Rightarrow (0;0)$ is not a maximum nor a minimum point.

$\left(\frac{8}{9};\frac{1}{3}\right)$ is the Maximum point.

$(0;1)$ and $(0;-1)$ are minimum points.



IV M4) $f(x;y;z) = x^2+y^2+z^2-xy+yz$.

$\nabla f(x;y;z) = (2x-y\,;\,2y-x+z\,;\,2z+y)$

$$H(f(x;y;z)) = \begin{Vmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{Vmatrix}.$$

$|H_1|:\ 2>0;\ 2>0;\ 2>0.$

$|H_2|:\ \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1=3>0;\ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1=3>0.$

$$|H_3| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\cdot\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 6-2=4>0.$$

The sequence of leading minors is $(+;+;+)\ \forall x,y,z$.

So the second order total differential d^2f is everywhere a positive definite quadratic form.