TASK MATHEMATICS for ECONOMIC APPLICATIONS 23/02/2016

I M 1) The complex numbers can written in trigonometric form as:

$i=\cos\left(\frac{\pi}{2}\right)+i\,\mathrm{sen}\left(\frac{\pi}{2}\right)=e^{i\frac{\pi}{2}}$, $\quad 1+i=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\,\mathrm{sen}\left(\frac{\pi}{4}\right)\right)=\sqrt{2}\,e^{i\frac{\pi}{4}}$ and $1-i=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\,\mathrm{sen}\left(-\frac{\pi}{4}\right)\right)=\sqrt{2}\,e^{-i\frac{\pi}{4}}$. So we have:

$$i^{15}\cdot\frac{(1+i)^5}{(1-i)^6}=\left(e^{i\frac{\pi}{2}}\right)^{15}\cdot\frac{\left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^5}{\left(\sqrt{2}\,e^{-i\frac{\pi}{4}}\right)^6}=e^{i\frac{15}{2}\pi}\cdot\frac{\left(\sqrt{2}\right)^5 e^{i\frac{5}{4}\pi}}{\left(\sqrt{2}\right)^6 e^{-i\frac{6}{4}\pi}}=\frac{1}{\sqrt{2}}\cdot e^{i\left(\frac{15}{2}\pi+\frac{5}{4}\pi+\frac{3}{2}\pi\right)}=$$

$$=\frac{\sqrt{2}}{2}\cdot e^{i\left(\frac{\pi}{4}+10\pi\right)}=\frac{\sqrt{2}}{2}\cdot e^{i\frac{\pi}{4}}=\frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{4}\right)+i\,\mathrm{sen}\left(\frac{\pi}{4}\right)\right)=\frac{1}{2}+\frac{1}{2}i\,.$$

I M 2) Conditions $f(1,0,1)=(1,0,1,0)$ and $f(1,-1,0)=(2,1,0,1)$ involve $\mathbb{A}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ and $\mathbb{A}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}2\\1\\0\\1\end{pmatrix}$ which gives the system:

$$\begin{cases}x_1+x_3=0\\y_1+y_3=1\\z_1+z_3=0\\x_1-x_2=1\\y_1-y_2=0\\z_1-z_2=1\end{cases}\Rightarrow\begin{cases}x_3=-x_1\\y_3=1-y_1\\z_3=-z_1\\x_2=x_1-1\\y_2=y_1\\z_2=z_1-1\end{cases}$$
. So the matrix $\mathbb{A}$ can be rewritten as:

$$\left\|\begin{matrix}1&-1&0\\x_1&x_1-1&-x_1\\y_1&y_1&1-y_1\\z_1&z_1-1&-z_1\end{matrix}\right\|$$
. For the condition on the Kernel remember that its dimension is given by the difference between the dimension of the domain of f and the dimension of the Image of f; so it follows that $\mathrm{Dim(Ker)}=1$ iff $\mathrm{Dim(Imm)}=\mathrm{Rank}(\mathbb{A})=2$. To calculate the Rank of the matrix we reduce it using elementary operations on the lines:

$$\left\|\begin{matrix}1&-1&0\\x_1&x_1-1&-x_1\\y_1&y_1&1-y_1\\z_1&z_1-1&-z_1\end{matrix}\right\|\overset{\substack{C_1+C_3\\C_2+C_3}}{\longrightarrow}\left\|\begin{matrix}1&-1&0\\0&-1&-x_1\\1&1&1-y_1\\0&-1&-z_1\end{matrix}\right\|\overset{\substack{R_3-R_1\\R_4-R_2}}{\longrightarrow}\left\|\begin{matrix}1&-1&0\\0&-1&-x_1\\0&2&1-y_1\\0&0&x_1-z_1\end{matrix}\right\|$$

$$\overset{R_3+2R_2}{\longrightarrow}\left\|\begin{matrix}1&-1&0\\0&-1&-x_1\\0&0&1-y_1-2x_1\\0&0&x_1-z_1\end{matrix}\right\|$$
and from the last matrix it follows that the Rank of $\mathbb{A}$ is equal to 2 iff $\begin{cases}1-y_1-2x_1=0\\x_1-z_1=0\end{cases}\Rightarrow\begin{cases}y_1=1-2x_1\\z_1=x_1\end{cases}$, and so the matrix $\mathbb{A}$ beco-

mes: $\left\|\begin{matrix} 1 & -1 & 0 \\ x_1 & x_1-1 & -x_1 \\ 1-2x_1 & 1-2x_1 & 2x_1 \\ x_1 & x_1-1 & -x_1 \end{matrix}\right\|$. Finally, since $(1,1,1)$ belongs to the Kernel of the linear map, we have: $\mathbb{A}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Rightarrow$

$$\left\|\begin{matrix} 1 & -1 & 0 \\ x_1 & x_1-1 & -x_1 \\ 1-2x_1 & 1-2x_1 & 2x_1 \\ x_1 & x_1-1 & -x_1 \end{matrix}\right\|\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\x_1-1\\2-2x_1\\x_1-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Rightarrow x_1=1\text{, and so:}$$

$\mathbb{A}=\left\|\begin{matrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \\ 1 & 0 & -1 \end{matrix}\right\|$. Since $(1,1,1)\in\text{Ker}(f)$ and $\text{Dim}(\text{Ker}(f))=1$, a basis for the Kernel is simply $\mathcal{B}_{\text{Ker}(f)}=\{(1,1,1)\}$.

I M 3) The characteristic polynomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda)=\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & k-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}=$

$$=(1-\lambda)\begin{vmatrix} k-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}+\begin{vmatrix} -1 & 1 \\ 0 & 1-\lambda \end{vmatrix}=$$
$$=(1-\lambda)\big(\lambda^2-(k+1)\lambda+k-1\big)-1+\lambda=$$
$$=(1-\lambda)\big(\lambda^2-(k+1)\lambda+k-2\big).$$

So $p_{\mathbb{A}}(1)=0$, and $\lambda=1$ is an eigenvalue of the matrix $\mathbb{A}$, $\forall k$.
If $q(\lambda)=\lambda^2-(k+1)\lambda+k-2$, it's easy to see that $q(1)=-2\neq 0$ and that $\Delta_q=(k+1)^2-4(k-2)=(k-1)^2+8>0,\ \forall k$.
So the matrix $\mathbb{A}$ has always three distinct real eigenvalues.

I M 4) To achieve the requested matrix the first step is to calculate the characteristic polynomial of $\mathbb{A}$:

$$p_{\mathbb{A}}(\lambda)=\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix}=(1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}+\begin{vmatrix} -1 & 1 \\ 0 & 1-\lambda \end{vmatrix}=$$
$$=(1-\lambda)\big(\lambda^2-3\lambda+1\big)-1+\lambda=-\lambda^3+4\lambda^2-3\lambda=\lambda\big(-\lambda^2+4\lambda-3\big)=$$

$=-\lambda(\lambda-1)(\lambda-3)$. The three eigenvalues of $\mathbb{A}$ are $\lambda_1=0,\lambda_2=1,\lambda_3=3$.
The second step is to find for any eigenvalue the corresponding eigenvector.
For $\lambda_1=0$ we must solve:

$$\|\mathbb{A}-0\cdot\mathbb{I}\|\cdot V=\mathbb{O}\Rightarrow\left\|\begin{matrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{matrix}\right\|\cdot\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow$$

$$\begin{cases} v_1-v_2=0 \\ -v_1+2v_2+v_3=0 \\ v_2+v_3=0 \end{cases}\Rightarrow\begin{cases} v_2=v_1 \\ v_3=-v_1 \end{cases}\Rightarrow V=\begin{pmatrix}v_1\\v_1\\-v_1\end{pmatrix}.$$

For $\lambda_2=1$ we must solve:

$$\|\mathbb{A}-1\cdot\mathbb{I}\|\cdot W=\mathbb{O}\Rightarrow\left\|\begin{matrix}0&-1&0\\-1&1&1\\0&1&0\end{matrix}\right\|\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow$$

$$\begin{cases}-w_2=0\\-w_1+w_2+w_3=0\\w_2=0\end{cases}\Rightarrow\begin{cases}w_2=0\\w_3=w_1\end{cases}\Rightarrow W=\begin{pmatrix}w_1\\0\\w_1\end{pmatrix}.$$

For $\lambda_3=3$ we must solve:

$$\|\mathbb{A}-3\cdot\mathbb{I}\|\cdot X=\mathbb{O}\Rightarrow\left\|\begin{matrix}-2&-1&0\\-1&-1&1\\0&1&-2\end{matrix}\right\|\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow$$

$$\begin{cases}-2x_1-x_2=0\\-x_1-x_2+x_3=0\\x_2-2x_3=0\end{cases}\Rightarrow\begin{cases}x_2=-2x_1\\x_3=-x_1\end{cases}\Rightarrow X=\begin{pmatrix}x_1\\-2x_1\\-x_1\end{pmatrix}.$$

The requested matrix is any matrix of the kind $\left\|\begin{matrix}v_1&w_1&x_1\\v_1&0&-2x_1\\-v_1&w_1&-x_1\end{matrix}\right\|$, with v_1, w_1 and x_1 all different from 0. If we want an orthogonal matrix, we have simply to determine the unit vectors of V, W and X, to get $U=\left\|\begin{matrix}\frac{\sqrt{3}}{3}&\frac{\sqrt{2}}{2}&\frac{\sqrt{6}}{6}\\\frac{\sqrt{3}}{3}&0&-\frac{\sqrt{6}}{3}\\-\frac{\sqrt{3}}{3}&\frac{\sqrt{2}}{2}&-\frac{\sqrt{6}}{6}\end{matrix}\right\|$.

Since the matrix $\mathbb{A}$ is symmetric, the three eigenvectors V, W and X are two by two orthogonal ($V\cdot W=0$, $V\cdot X=0$ and $W\cdot X=0$).

II M 1) It's easy to verify that at point $(-1,1,-1)$ the equations of the system are satisfied. The Jacobian matrix of the functions is:

$$J_{f,g}=\left\|\begin{matrix}4x^3e^{x^4-y^4}-2yz&-4y^3e^{x^4-y^4}-4y^3e^{z^4-y^4}-2xz&4z^3e^{z^4-y^4}-2xy\\3x^2+3z&-3y^2&3z^2+3x\end{matrix}\right\|\quad\text{with}$$

$$J_{f,g}(-1,1,-1)=\left\|\begin{matrix}-2&-10&-2\\0&-3&0\end{matrix}\right\|.$$

Since $\left|\begin{matrix}-10&-2\\-3&0\end{matrix}\right|=-6\neq0$ the function $x\to(y(x),z(x))$ exists and its derivatives are given by:

$$\begin{pmatrix}y'(-1)\\z'(-1)\end{pmatrix}=-\left(J_{f,g}(-1,1,-1)\Big|_{(y,z)}\right)^{-1}\cdot\left(J_{f,g}(-1,1,-1)\Big|_x\right)=$$

$$=-\left\|\begin{matrix}-10&-2\\-3&0\end{matrix}\right\|^{-1}\cdot\left\|\begin{matrix}-2\\0\end{matrix}\right\|=\frac{1}{6}\cdot\left\|\begin{matrix}0&2\\3&-10\end{matrix}\right\|\cdot\left\|\begin{matrix}-2\\0\end{matrix}\right\|=\left\|\begin{matrix}0\\-1\end{matrix}\right\|\quad\text{or}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\left|\begin{matrix}-2&-2\\0&0\end{matrix}\right|}{-6}=0\ \text{and}\ \frac{\mathrm{d}z}{\mathrm{d}x}=-\frac{\left|\begin{matrix}-10&-2\\-3&0\end{matrix}\right|}{-6}=-1.$$

II M 2) Since the function $f(x,y)=e^{\alpha(x-y)}$ is clearly twice differentiable, we have:

$$D_vf(0,0)=\nabla f(0,0)\cdot\frac{v}{\|v\|}\ \text{and}\ D^2_{v,-v}f(0,0)=\left(\frac{v}{\|v\|}\right)\cdot\mathbb{H}(f(0,0))\cdot\left(-\frac{v}{\|v\|}\right)^T.$$

Hence we have: $\nabla f(x,y)=\left(\alpha e^{\alpha(x-y)},-e^{\alpha(x-y)}\right)=\alpha\cdot f(x,y)\cdot(1,-1)$,

$$\nabla f(0,0)=\alpha\cdot(1,\,-1)\Rightarrow D_v f(0,0)=\alpha\cdot(1,\,-1)\cdot\left(\frac{\sqrt{2}}{2},\,-\frac{\sqrt{2}}{2}\right)=\sqrt{2}\alpha\,.$$

Then:

$$\mathbb{H}(f(x,y))=\left\|\begin{matrix}\alpha^2 e^{\alpha(x-y)} & -\alpha^2 e^{\alpha(x-y)}\\ -\alpha^2 e^{\alpha(x-y)} & \alpha^2 e^{\alpha(x-y)}\end{matrix}\right\|=\alpha^2\cdot f(x,y)\cdot\left\|\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right\|,$$

$\mathbb{H}(f(0,0))=\alpha^2\cdot\left\|\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right\|$ and we get

$$D^2_{v,-v}f(0,0)=\left\|\begin{matrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{matrix}\right\|\cdot\left(\alpha^2\cdot\left\|\begin{matrix}1 & -1\\ -1 & 1\end{matrix}\right\|\right)\cdot\left\|\begin{matrix}-\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}\end{matrix}\right\|=-2\alpha^2\,.$$

From $D_v f(0,0)+D^2_{v,-v}f(0,0)=\dfrac{1}{4}$ we get:

$$-2\alpha^2+\sqrt{2}\alpha=\frac{1}{4}\Rightarrow 8\alpha^2-4\sqrt{2}\alpha+1=0\Rightarrow\left(2\sqrt{2}\alpha-1\right)^2=0\Rightarrow\alpha=\frac{1}{2\sqrt{2}}\,.$$

II M 3) The Lagrangian function of the problem is:

$\Lambda(x,y,z,\lambda)=x+y+z-\lambda\left(x^2+2y^2+3z^2-66\right)$.

Then $\nabla\Lambda=\left(1-2\lambda x,1-4\lambda y,1-6\lambda z,\,-\left(x^2+2y^2+3z^2-66\right)\right)$.

I Order Conditions:

$$\begin{cases}1-2\lambda x=0\\ 1-4\lambda y=0\\ 1-6\lambda z=0\\ x^2+2y^2+3z^2=66\end{cases}\Rightarrow\begin{cases}x=1/(2\lambda)\\ y=1/(4\lambda)\\ z=1/(6\lambda)\\ 11/(24\lambda^2)=66\end{cases}\Rightarrow\begin{cases}x=\pm 6\\ y=\pm 3\\ z=\pm 2\\ \lambda=\pm 1/12\end{cases}.$$

The problem has two constrained critical points: $P_1=(6,3,2),P_2=(-6,\,-3,\,-2)$.

II Order Conditions:

$$\overline{\mathbb{H}}(\Lambda)=\left\|\begin{matrix}0 & -2x & -4y & -6z\\ -2x & -2\lambda & 0 & 0\\ -4y & 0 & -4\lambda & 0\\ -6z & 0 & 0 & -6\lambda\end{matrix}\right\|$$

. Since our optimization problem has 3 variables and 1 constraint, we must consider 2 principal minors, the thirth and the fourth:

$$\left|\overline{\mathbb{H}}_3\right|=\left|\begin{matrix}0 & -2x & -4y\\ -2x & -2\lambda & 0\\ -4y & 0 & -4\lambda\end{matrix}\right|=2x\cdot\left|\begin{matrix}-2x & 0\\ -4y & -4\lambda\end{matrix}\right|-4y\cdot\left|\begin{matrix}-2x & -2\lambda\\ -4y & 0\end{matrix}\right|=$$

$$=16\lambda x^2+32\lambda y^2=16\lambda\left(x^2+2y^2\right);$$

$$\left|\overline{\mathbb{H}}_4\right|=\left|\begin{matrix}0 & -2x & -4y & -6z\\ -2x & -2\lambda & 0 & 0\\ -4y & 0 & -4\lambda & 0\\ -6z & 0 & 0 & -6\lambda\end{matrix}\right|=$$

$$=6z\cdot\left|\begin{matrix}-2x & -4y & -6z\\ -2\lambda & 0 & 0\\ 0 & -4\lambda & 0\end{matrix}\right|-6\lambda\cdot\left|\begin{matrix}0 & -2x & -4y\\ -2x & -2\lambda & 0\\ -4y & 0 & -4\lambda\end{matrix}\right|=$$

$$=6z\cdot 2\lambda\cdot\left|\begin{matrix}-4y & -6z\\ -4\lambda & 0\end{matrix}\right|-6\lambda\cdot\left(2x\cdot\left|\begin{matrix}-2x & -4y\\ 0 & -4\lambda\end{matrix}\right|-4y\cdot\left|\begin{matrix}-2x & -4y\\ -2\lambda & 0\end{matrix}\right|\right)=$$

$$=12\lambda z(-24\lambda z)-12\lambda x(8\lambda x)+24\lambda y(-8\lambda y)=-96\lambda^2\left(x^2+2y^2+3z^2\right).$$

Substituting we get:

$\left|\overline{\mathbb{H}}_3(P_1)\right|>0,\left|\overline{\mathbb{H}}_4(P_1)\right|<0\Rightarrow P_1$ is a Maximum point with $f(P_1)=11$;

$\left|\overline{\mathbb{H}}_3(P_2)\right|<0,\left|\overline{\mathbb{H}}_4(P_2)\right|<0\Rightarrow P_2$ is a Minimum point with $f(P_2)=-11$.

II M 4) The second order total differential d^2f of the function $f(x,y)$ is equal to:

$$\|dx\ dy\| \cdot \mathbb{H}f(x,y) \cdot \left\| \begin{matrix} dx \\ dy \end{matrix} \right\| = \|dx\ dy\| \cdot \left\| \begin{matrix} 9e^{3x-y} & -3e^{3x-y} \\ -3e^{3x-y} & e^{3x-y} \end{matrix} \right\| \cdot \left\| \begin{matrix} dx \\ dy \end{matrix} \right\| =$$

$$= e^{3x-y} \cdot \|dx\ dy\| \cdot \left\| \begin{matrix} 9 & -3 \\ -3 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} dx \\ dy \end{matrix} \right\| = e^{3x-y}\big(9\,d^2x - 6\,dx\,dy + d^2y\big) =$$

$= e^{3x-y}(3\,dx - dy)^2 \geq 0$, for any couple of increments (dx, dy). So d^2f is always a positive semidefinite quadratic form.