TASK MATHEMATICS for ECONOMIC APPLICATIONS 19/03/2016

I M 1) Putting the complex number in trigonometric form we get:

$$z = i^{19} - 3i^6 - i^{18} + 3i^3 = i^3 - 3i^2 - i^2 + 3i^3 =$$

$$= 4i^3 - 4i^2 = 4 - 4i = 4\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}\left(\cos\frac{7}{4}\pi + i\,\text{sen}\frac{7}{4}\pi\right).$$

The fourth roots of z are $\sqrt[4]{z} = \sqrt[4]{4\sqrt{2}\left(\cos\frac{7}{4}\pi + i\,\text{sen}\frac{7}{4}\pi\right)} =$

$$= \sqrt[4]{4\sqrt{2}}\left(\cos\left(\frac{7\pi/4 + 2k\pi}{4}\right) + i\,\text{sen}\left(\frac{7\pi/4 + 2k\pi}{4}\right)\right) =$$

$$= \sqrt[4]{4\sqrt{2}}\left(\cos\left(\frac{7}{16}\pi + k\frac{\pi}{2}\right) + i\,\text{sen}\left(\frac{7}{16}\pi + k\frac{\pi}{2}\right)\right),\ k = 0, 1, 2, 3.$$

The imaginary part of $\sqrt[4]{z}$ is positive iff $\text{sen}\left(\frac{7}{16}\pi + k\frac{\pi}{2}\right) > 0$ and this is true iff $0 < \frac{7}{16}\pi + k\frac{\pi}{2} < \pi \Rightarrow -\frac{7}{8} < k < \frac{9}{8} \Rightarrow k = 0, 1$. The requested roots are:

$$z_0 = \sqrt[4]{4\sqrt{2}}\left(\cos\frac{7}{16}\pi + i\,\text{sen}\frac{7}{16}\pi\right) =$$

$$= \frac{\sqrt[4]{4\sqrt{2}}}{2}\left(\sqrt{2 - \sqrt{2 + \sqrt{2}}} + i\sqrt{2 + \sqrt{2 + \sqrt{2}}}\right);$$

$$z_1 = \sqrt[4]{4\sqrt{2}}\left(\cos\frac{15}{16}\pi + i\,\text{sen}\frac{15}{16}\pi\right) =$$

$$= \frac{\sqrt[4]{4\sqrt{2}}}{2}\left(-\sqrt{2 + \sqrt{2 + \sqrt{2}}} + i\sqrt{2 - \sqrt{2 + \sqrt{2}}}\right).$$

I M 2) By the Sylvester Theorem $\text{Dim(Ker)} + \text{Dim(Imm)} = \text{Dim}(\mathbb{R}^4) = 4$.
If $\text{Dim(Ker)} = 2$ trivialy follows $\text{Dim(Imm)} = \text{Rank}(\mathbb{A}) = 2$.
To find m and k such that $\text{Rank}(\mathbb{A})$ is 2 we reduce matrix $\mathbb{A}$ by elementary operations on the lines:

$$\left\|\begin{matrix} 1 & 1 & 2 & -2 \\ 1 & 2 & 1 & 1 \\ 3 & 2 & m & k \end{matrix}\right\| \overset{\substack{R_2 - R_1 \\ R_3 - 3R_1}}{\Rightarrow} \left\|\begin{matrix} 1 & 1 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & m-6 & k+6 \end{matrix}\right\| \overset{R_3 + R_2}{\Rightarrow}$$

$\Rightarrow \left\|\begin{matrix} 1 & 1 & 2 & -2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & m-7 & k+9 \end{matrix}\right\|$ and from the last matrix it follows that $\text{Rank}(\mathbb{A}) = 2$ iff $(m - 7 = 0 \text{ and } k + 9 = 0) \Rightarrow (m = 7 \text{ and } k = -9)$.

So the matrix $\mathbb{A}$ is equal to $\left\|\begin{matrix} 1 & 1 & 2 & -2 \\ 1 & 2 & 1 & 1 \\ 3 & 2 & 7 & -9 \end{matrix}\right\|$. To find a basis for the Kernel remember that $\mathbb{X}$ belongs to the $\text{Ker}(f)$ if $\mathbb{A} \cdot \mathbb{X} = \mathbb{O}$, that in system form is:

$$\begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = 0 \\ x_1 + 2x_2 + x_3 + x_4 = 0 \\ 3x_1 + 2x_2 + 7x_3 - 9x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = 0 \\ x_2 - x_3 + 3x_4 = 0 \\ -x_2 + x_3 - 3x_4 = 0 \end{cases} \Rightarrow$$

$\Rightarrow \begin{cases} x_1 = -3x_3 + 5x_4 \\ x_2 = x_3 - 3x_4 \end{cases}$, and so a generic element of Ker(f) is:
$\mathbb{X} = (-3x_3 + 5x_4, x_3 - 3x_4, x_3, x_4) = x_3(-3,1,1,0) + x_4(5,-3,0,1)$ and a basis for Ker(f) is $\mathcal{B}_{\text{Ker}(f)} = \{(-3,1,1,0),(5,-3,0,1)\}$.
To find a basis for the Immage remember that $\mathbb{Y}$ belongs to Imm(f) if $\mathbb{A} \cdot \mathbb{X} = \mathbb{Y}$, that in system form is:

$$\begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = y_1 \\ x_1 + 2x_2 + x_3 + x_4 = y_2 \\ 3x_1 + 2x_2 + 7x_3 - 9x_4 = y_3 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = y_1 \\ x_2 - x_3 + 3x_4 = y_2 - y_1 \\ -x_2 + x_3 - 3x_4 = y_3 - 3y_1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = y_1 \\ x_2 - x_3 + 3x_4 = y_2 - y_1 \\ 0 = y_3 + y_2 - 4y_1 \end{cases}.$$

So a generic element of Imm(f) is a vector $\mathbb{Y} = (y_1, y_2, y_3)$ with $y_3 + y_2 - 4y_1 = 0$ or $y_3 = 4y_1 - y_2$; $\mathbb{Y} = (y_1, y_2, 4y_1 - y_2) = y_1(1,0,4) + y_2(0,1,-1)$ and a basis for Imm(f) is $\mathcal{B}_{\text{Imm}(f)} = \{(1,0,4),(0,1,-1)\}$.

I M 3) To find the characteristic polynomial of $\mathbb{A}$ we calculate:

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 3-\lambda & 2 & 1 \\ 1 & 4-\lambda & k \\ 1 & 2 & -3-\lambda \end{vmatrix} =$$

$$= \begin{vmatrix} 2-\lambda & 0 & 4+\lambda \\ 1 & 4-\lambda & k \\ 1 & 2 & -3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 4-\lambda & k \\ 2 & -3-\lambda \end{vmatrix} + (4+\lambda)(2-4+\lambda) =$$

$$= (2-\lambda)\left(\lambda^2 - \lambda - 12 - 2k\right) - (4+\lambda)(2-\lambda) =$$

$$= (2-\lambda)\left(\lambda^2 - 2\lambda - 2k - 16\right).$$

Note that $\forall k,\ p_{\mathbb{A}}(2) = 0$, this implies that $\lambda = 2$ is an eigenvalue of the matrix $\mathbb{A}$, $\forall k$.
If $q(\lambda) = \lambda^2 - 2\lambda - 2k - 16$, $\lambda = 2$ is a multiple eingenvalue for $\mathbb{A}$ iff:
$q(2) = 4 - 4 - 2k - 16 = 0 \Rightarrow k = -8$. In this case the three eingevalues for $\mathbb{A}$ are $\lambda_1 = 0$ and $\lambda_{2,3} = 2$. To study the diagonalizability of $\mathbb{A}$ we must check the dimension of the eingespace associated to $\lambda = 2$, that is:

$$3 - \text{Rank}(\mathbb{A} - 2\mathbb{I}) = 3 - \text{Rank}\left(\begin{Vmatrix} 1 & 2 & 1 \\ 1 & 2 & -8 \\ 1 & 2 & -5 \end{Vmatrix}\right) =$$

$$= 3 - \text{Rank}\left(\begin{Vmatrix} 1 & 2 & 1 \\ 0 & 0 & -9 \\ 0 & 0 & -6 \end{Vmatrix}\right) = 3 - 2 = 1.$$

Matrix $\mathbb{A}$ isn't diagonalizable since the algebraic multiplicity of $\lambda = 2$ is greater than its geometric multiplicity.

I M 4) Two similar matrices have equal characteristic polynomials, and so we get:

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1 \text{ and}$$

$$p_{\mathbb{B}}(\lambda) = |\mathbb{B} - \lambda\mathbb{I}| = \begin{vmatrix} -\lambda & m \\ 1 & k-\lambda \end{vmatrix} = \lambda^2 - k\lambda - m.$$

The two polynomials are equal iff $k = 4$ and $m = -1$.
To find a matrix $\mathbb{P} = \begin{Vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{Vmatrix}$ we must satisfy $\mathbb{A} \cdot \mathbb{P} = \mathbb{P} \cdot \mathbb{B} \Rightarrow$

$$\Rightarrow \begin{Vmatrix} 1 & 2 \\ 1 & 3 \end{Vmatrix} \cdot \begin{Vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{Vmatrix} = \begin{Vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 0 & -1 \\ 1 & 4 \end{Vmatrix} \Rightarrow$$

$\Rightarrow \begin{Vmatrix} p_{11}+2p_{21} & p_{12}+2p_{22} \\ p_{11}+3p_{21} & p_{12}+3p_{22} \end{Vmatrix} = \begin{Vmatrix} p_{12} & 4p_{12}-p_{11} \\ p_{22} & 4p_{22}-p_{21} \end{Vmatrix}$. In system form we have:

$$\begin{cases} p_{11}+2p_{21}=p_{12} \\ p_{12}+2p_{22}=4p_{12}-p_{11} \\ p_{11}+3p_{21}=p_{22} \\ p_{12}+3p_{22}=4p_{22}-p_{21} \end{cases} \Rightarrow \begin{cases} p_{11}+2p_{21}=p_{12} \\ p_{11}+2p_{22}=3p_{12} \\ p_{11}+3p_{21}=p_{22} \\ p_{12}+p_{21}=p_{22} \end{cases} \Rightarrow \begin{cases} p_{11}=2p_{21}-p_{12} \\ p_{22}=p_{12}+p_{21} \end{cases}.$$

Choosing, for instance, $p_{12}=2$ and $p_{21}=1$, one possible matrix $\mathbb{P}$ is $\begin{Vmatrix} 0 & 2 \\ 1 & 3 \end{Vmatrix}$.

II M 1) For any pair of unit vectors (v,w), $D^2_{v,w}f(x,y) = v \cdot \mathbb{H}(f) \cdot w^T$ where $\mathbb{H}(f)$ is the Hessian matrix of f. From this it follows easily that $D^2_{v,v}f(x,y) = D^2_{-v,-v}f(x,y)$. So $D^2_{v,v}f(x,y) + D^2_{-v,-v}f(x,y) = 0$ iff $D^2_{v,v}f(x,y)=0$. Since:

$\nabla f = \left(2xe^{x^2-y^2},\ -2ye^{x^2-y^2}\right)$ and $\mathbb{H}(f) = \begin{Vmatrix} 2(2x^2+1)e^{x^2-y^2} & -4xye^{x^2-y^2} \\ -4xye^{x^2-y^2} & 2(2y^2-1)e^{x^2-y^2} \end{Vmatrix}$

we have:

$$D^2_{v,v}f(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \begin{Vmatrix} 2(2x^2+1)e^{x^2-y^2} & -4xye^{x^2-y^2} \\ -4xye^{x^2-y^2} & 2(2y^2-1)e^{x^2-y^2} \end{Vmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$= \left(2x^2+1\right)e^{x^2-y^2} - 4xye^{x^2-y^2} + \left(2y^2-1\right)e^{x^2-y^2} = 2(x-y)^2e^{x^2-y^2}.$$

So we have $D^2_{v,v}f(x,y)=0$ iff $x-y=0$.

II M 2) The equation is satisfied at point $P=(0,1)$, since $f(x,y) = \log\left(x^2+y^2\right) - xy \Rightarrow \log(1) - 0 = 0$. Then $\nabla f = \left(\frac{2x}{x^2+y^2} - y, \frac{2y}{x^2+y^2} - x\right)$ with $\nabla f(0,1) = (-1,2)$, and so

$$y'(0) = -\frac{f'_x(0,1)}{f'_y(0,1)} = \frac{1}{2}.$$

For the second order derivative we have: $y'' = -\dfrac{f''_{xx} + 2f''_{xy}\cdot y' + f''_{yy}\cdot (y')^2}{f'_y}$,

$\mathbb{H}(f) = \begin{Vmatrix} \frac{2(y^2-x^2)}{(x^2+y^2)^2} & \frac{-4xy}{(x^2+y^2)^2} - 1 \\ \frac{-4xy}{(x^2+y^2)^2} - 1 & \frac{2(x^2-y^2)}{(x^2+y^2)^2} \end{Vmatrix}$, $\mathbb{H}(f(0,1)) = \begin{Vmatrix} 2 & -1 \\ -1 & -2 \end{Vmatrix}$ and so

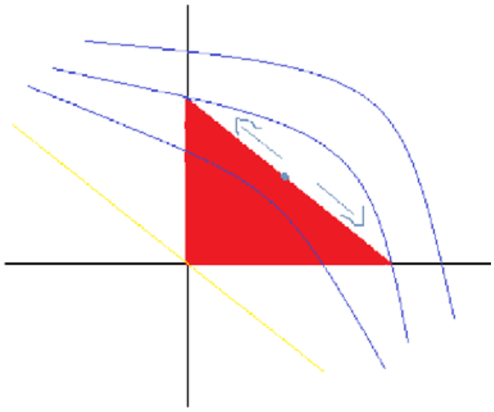
$$y''(0) = -\frac{2 + 2(-1)\cdot\frac{1}{2} + (-2)\cdot\left(\frac{1}{2}\right)^2}{2} = -\frac{1}{4}.$$

II M 3) From $y = f(x_1,x_2)$ and $(x_1,x_2) = g(t_1,t_2,t_3)$, by the chain rule (derivative of a composite function) we get: $\dfrac{\partial(y)}{\partial(t_1,t_2,t_3)} = \dfrac{\partial(y)}{\partial(x_1,x_2)} \cdot \dfrac{\partial(x_1,x_2)}{\partial(t_1,t_2,t_3)}$ or

$\begin{Vmatrix} 1 & 3 & 4 \end{Vmatrix} = \begin{Vmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} \end{Vmatrix} \cdot \begin{Vmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{Vmatrix}$ that in system form gives:

$$\begin{cases} \frac{\partial y}{\partial x_1} = 1 \\ -\frac{\partial y}{\partial x_1} + 2\frac{\partial y}{\partial x_2} = 3 \\ 2\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} = 4 \end{cases} \Rightarrow \begin{cases} \frac{\partial y}{\partial x_1} = 1 \\ \frac{\partial y}{\partial x_2} = 2 \end{cases} \Rightarrow \frac{\partial(y)}{\partial(x_1, x_2)} = \|1 \quad 2\| .$$

II M 4) The triangle T of vertexes $(0,0)$, $(1,0)$ and $(0,1)$ is red-drawed in the figure:



T is a compact set and the function f is continuous so it admits absolute maximum and minimum on T; $\forall(x,y) \in T$, x and y are both not negative and trivially follows that $\min(f) = 0$ at point $(0,0)$. For the maximum we observe that when at least one between x and y increases, then f increases, thus $\text{Max}(f)$ must be searched on the upper border of T having equation $y = 1 - x$. If we define:

$g(x) = f(x, 1-x) = x^3 + (1-x)^3$, we get:

$g'(x) = 3x^2 - 3(1-x)^2 = 3\left(x^2 - 1 - x^2 + 2x\right) = 3(2x-1)$ and

$g'(x) \geq 0$ iff $3(2x-1) \geq 0 \Rightarrow x \geq 1/2$.

Function g increases from point $(1/2, 1/2)$ to the vertexes $(1,0)$ and $(0,1)$ of T (grey arrows in the figure): by the simmetry of f we conclude that $\text{Max}(f) = 1$ at points $(1,0)$ and $(0,1)$. Point $(1/2, 1/2)$ is a relative minimum point.

In the figure there are drawn zero level curve (yellow) and positive level curves (blue).