TASK of MATHEMATICS for ECONOMIC APPLICATIONS 04/06/2016

I M 1) If $k \neq -2i$, relation $\dfrac{(2+i)^2-(1-i)^2}{k+2i}=i$ can be written as
$(2+i)^2-(1-i)^2=i(k+2i)$ equivalent to
$((2+i)-(1-i))((2+i)+(1-i))=ik-2 \Rightarrow 3(1+2i)=ik-2 \Rightarrow$
$k=\dfrac{1}{i}\cdot(5+6i)=-i\cdot(5+6i)=6-5i$.

I M 2) An element (x_1,x_2,x_3,x_4) belongs to the Kernel of f if:
$f(x_1,x_2,x_3,x_4)=(0,0,0)$ that is, in system form:
$\begin{cases} x_1+x_2=0 \\ x_2+x_3=0 \\ x_3+x_4=0 \end{cases} \Rightarrow \begin{cases} x_2=-x_1 \\ x_3=-x_2=x_1 \\ x_4=-x_3=-x_1 \end{cases}$. So $\text{Dim}(\text{Ker}(f))=1$ and every element of $\text{Ker}(f)$ is of the form $\mathbb{X}=(x_1,-x_1,x_1,-x_1)=k\cdot(1,-1,1,-1)$ and a basis for $\text{Ker}(f)$ is $\mathcal{B}_{\text{Ker}(f)}=\{(1,-1,1,-1)\}$.
From Sylvester Theorem: $\text{Dim}(\text{Ker}(f))+\text{Dim}(\text{Imm}(f))=\text{Dim}(\mathbb{R}^4)=4$.
Since $\text{Dim}(\text{Ker}(f))=1$ we have $\text{Dim}(\text{Imm})=4-1=3$ and so $\text{Imm}(f)=\mathbb{R}^3$ and a basis for $\text{Imm}(f)$ is simply given by the standard basis:
$\mathcal{B}_{\text{Imm}(f)}=\{e_1,e_2,e_3\}=\{(1,0,0),(0,1,0),(0,0,1)\}$.

I M 3) Since the vector $\mathbb{X}$ has coordinates $(1,-2,1)$ in the basis $\mathbb{W}$, it has coordinates (α,β,γ) in the basis $\mathbb{V}$ if

$$\alpha\begin{pmatrix}1\\0\\1\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\end{pmatrix}=1\begin{pmatrix}1\\1\\1\end{pmatrix}-2\begin{pmatrix}1\\-1\\1\end{pmatrix}+1\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

that in system form gives:

$$\begin{cases}\alpha=0\\ \beta=4\\ \alpha+\gamma=-2\end{cases} \Rightarrow \begin{cases}\alpha=0\\ \beta=4\\ \gamma=-2\end{cases}.$$

So in the basis $\mathbb{V}$ the vector $\mathbb{X}$ has coordinates $(0,4,-2)$.

I M 4) The characteristic polinomial of $\mathbb{A}$ is

$$p_{\mathbb{A}}(\lambda)=|\mathbb{A}-\lambda\mathbb{I}|=\begin{vmatrix}3-\lambda & 1 & 1\\ 2 & 4-\lambda & 2\\ 1 & m & k-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda & 1 & 1\\ 0 & 4-\lambda & 2\\ 1-k+\lambda & m & k-\lambda\end{vmatrix}=$$

$$=(2-\lambda)\begin{vmatrix}4-\lambda & 2\\ m & k-\lambda\end{vmatrix}+(1-k+\lambda)\begin{vmatrix}1 & 1\\ 4-\lambda & 2\end{vmatrix}=$$

$$=(2-\lambda)\left(\lambda^2-(4+k)\lambda+4k-2m\right)+(1-k+\lambda)(2-4+\lambda)=$$

$$=(2-\lambda)\left(\lambda^2-(4+k)\lambda+4k-2m\right)+(1-k+\lambda)(\lambda-2)=$$

$$=(2-\lambda)\left(\lambda^2-(4+k)\lambda+4k-2m-1+k-\lambda\right)=$$

$$=(2-\lambda)\left(\lambda^2-(5+k)\lambda+5k-2m-1\right).$$

So $p_{\mathbb{A}}(2)=0$ and $\lambda=2$ is an eigenvalue of the matrix $\mathbb{A}, \forall(k,m)$.
If $q(\lambda)=\lambda^2-(5+k)\lambda+5k-2m-1$, $\lambda=2$ is a multiple eingenvalue for $\mathbb{A}$ iff $q(2)=2m-3k+7=0$ i.e. iff $m=\dfrac{3k-7}{2}$.
In such case the characteristic polynomial $p_{\mathbb{A}}(\lambda)$ can be factorized as:

$p_{\mathbb{A}}(\lambda) = -(\lambda-2)^2(\lambda-(3+k))$ and the three eingevalues for $\mathbb{A}$ are $\lambda_{1,2}=2$ and $\lambda_3 = 3+k$. If $k=-1$ (and so $m=-5$) it follows that $\lambda_{1,2,3}=2$. So we have: if $m=\dfrac{3k-7}{2}$ and $k\neq -1 \Rightarrow m_2^a = 2$ while if $m=-5$ and $k=-1 \Rightarrow m_2^a=3$.
To study the diagonalizability of $\mathbb{A}$ when $\lambda=2$ is a multiple eingenvalue, we must check the dimension of the eingespace associated to $\lambda=2$ if $m=\dfrac{3k-7}{2}$, that is:

$$\text{Order}(\mathbb{A}) - \text{Rank}(\mathbb{A}-2\mathbb{I}) = 3 - \text{Rank}\left(\left\|\begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & \frac{3k-7}{2} & k-2 \end{matrix}\right\|\right) =$$

$$= 3 - \text{Rank}\left(\left\|\begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & \frac{3k-9}{2} & k-3 \end{matrix}\right\|\right) = \begin{cases} 1 & \text{if } k \neq 3 \\ 2 & \text{if } k=3 \end{cases}.$$

The matrix $\mathbb{A}$ is diagonalizable when $m=1$ and $k=3$, isn't diagonalizable when $m=\dfrac{3k-7}{2}$ and $k\neq 3$.

II M 1) For a two times differentiable function $f(x,y)$ at stationary point P_0 we have:

$$D^2_{e_1,-e_2} f(P_0) = e_1 \cdot \mathbb{H}(f(P_0)) \cdot (-e_2)^T = (1 \quad 0) \cdot \left\|\begin{matrix} f''_{xx}(P_0) & f''_{xy}(P_0) \\ f''_{xy}(P_0) & f''_{yy}(P_0) \end{matrix}\right\| \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} =$$

$$= -(1 \quad 0) \cdot \begin{pmatrix} f''_{xy}(P_0) \\ f''_{yy}(P_0) \end{pmatrix} = -f''_{xy}(P_0) = 2 \Rightarrow f''_{xy}(P_0) = -2.$$ In the same way:

$$D^2_{-e_1,e_1} f(P_0) = (-1 \quad 0) \cdot \left\|\begin{matrix} f''_{xx}(P_0) & f''_{xy}(P_0) \\ f''_{xy}(P_0) & f''_{yy}(P_0) \end{matrix}\right\| \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -f''_{xx}(P_0)$$ and

$D^2_{-e_1,e_1} f(P_0) = 3$ implies $f''_{xx}(P_0) = -3$. Finally

$$D^2_{e_2,e_2} f(P_0) = (0 \quad 1) \cdot \left\|\begin{matrix} f''_{xx}(P_0) & f''_{xy}(P_0) \\ f''_{xy}(P_0) & f''_{yy}(P_0) \end{matrix}\right\| \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = f''_{yy}(P_0)$$ and

$D^2_{e_2,e_2} f(P_0) = -2$ implies $f''_{yy}(P_0) = -2$.

So the Hessian matrix of f at point P_0 is $\mathbb{H}(f(P_0)) = \left\|\begin{matrix} -3 & -2 \\ -2 & -2 \end{matrix}\right\|$.

To check the nature of P_0 we have: $\begin{cases} |\mathbb{H}_1| = -3 < 0;\ -2 < 0 \\ |\mathbb{H}_2| = 6-4 > 0 \end{cases}$ and so P_0 is a maximum point.

II M 2) First step (we calculate functions f and g at point $(1,0,1)$):

$\begin{cases} f(1,0,1) = 1-1-0 = 0 \\ g(1,0,1) = 1\,e^0 - 1\,e^0 - 0 = 0 \end{cases}$, so the conditions are satisfied.

Second step (we calculate the Jacobian matrix of f and g):

$$J = \frac{\partial(f,g)}{\partial(x,y,z)} = \left\|\begin{matrix} 1 & -(1+y)\,e^y & -1 \\ (1-x)e^{z-x} - z\,e^{x-z} & -2y & xe^{z-x} - (1-z)e^{x-z} \end{matrix}\right\|.$$

Third step (we calculate the Jacobian matrix of f and g at point $(1,0,1)$):

$$J(1,0,1) = \left\|\begin{matrix} 1 & -1 & -1 \\ -1 & 0 & 1 \end{matrix}\right\|.$$

Since $\left| J(1,0,1)\Big|_{(x,z)} \right| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$, while

$\left| J(1,0,1)\Big|_{(x,y)} \right| = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = -1$ and

$\left| J(1,0,1)\big|_{(y,z)} \right| = \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} = -1$, with conditions $\begin{cases} f(1,0,1)=0 \\ g(1,0,1)=0 \end{cases}$ we can define a function $z \mapsto (x(z), y(z))$ or a function $x \mapsto (y(x), z(x))$, both in a neighbourhood of point 1.

Last step (we calculate the derivatives of the implicit function).

For function $z \mapsto (x(z), y(z))$:

$$\begin{pmatrix} x'(1) \\ y'(1) \end{pmatrix} = -\left(J(1,0,1)\big|_{(x,y)} \right)^{-1} \cdot \left(J(1,0,1)\big|_{(z)} \right) =$$

$$= -\begin{Vmatrix} 1 & -1 \\ -1 & 0 \end{Vmatrix}^{-1} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{Vmatrix} 0 & 1 \\ 1 & 1 \end{Vmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

For function $x \mapsto (y(x), z(x))$:

$$\begin{pmatrix} y'(1) \\ z'(1) \end{pmatrix} = -\left(J(1,0,1)\big|_{(y,z)} \right)^{-1} \cdot \left(J(1,0,1)\big|_{(x)} \right) =$$

$$= -\begin{Vmatrix} -1 & -1 \\ 0 & 1 \end{Vmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\begin{Vmatrix} 1 & 1 \\ 0 & -1 \end{Vmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

II M 3) From $f(x,y,z) = x^3 - 8y^3 - z^3 + 3x^2 - 24yz$ we get

$$\nabla f = \left(3x^2 + 6x, -24y^2 - 24z, -3z^2 - 24y\right), \ \mathbb{H}f = \begin{Vmatrix} 6x+6 & 0 & 0 \\ 0 & -48y & -24 \\ 0 & -24 & -6z \end{Vmatrix}.$$

First order conditions: $\nabla f = \mathbb{O}$

$$\Rightarrow \begin{cases} 3x^2 + 6x = 0 \\ -24y^2 - 24z = 0 \\ -3z^2 - 24y = 0 \end{cases} \Rightarrow \begin{cases} 3x(x+2) = 0 \\ -24(y^2 + z) = 0 \\ -3(z^2 + 8y) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \vee x = -2 \\ z = -y^2 \\ y^4 + 8y = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 0 \vee x = -2 \\ z = -y^2 \\ y(y^3 + 8) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \vee x = -2 \\ z = -y^2 \\ y = 0 \vee y = -2 \end{cases}.$$

The function f has four stationary points:

$P_1 = (0,0,0)$, $P_2 = (-2,0,0)$, $P_3 = (0,-2,-4)$ and $P_4 = (-2,-2,-4)$.

Second order conditions:

$\mathbb{H}f(P_1) = \begin{Vmatrix} 6 & 0 & 0 \\ 0 & 0 & -24 \\ 0 & -24 & 0 \end{Vmatrix}$, $\mathbb{H}f(P_2) = \begin{Vmatrix} -6 & 0 & 0 \\ 0 & 0 & -24 \\ 0 & -24 & 0 \end{Vmatrix}$, both having a second order principal minor $\mathbb{H}_2 = \begin{Vmatrix} 0 & -24 \\ -24 & 0 \end{Vmatrix}$ with a negative determinant equal to $-576 < 0$, and so P_1 and P_2 are saddle points.

$\mathbb{H}f(P_3) = \begin{Vmatrix} 6 & 0 & 0 \\ 0 & 96 & -24 \\ 0 & -24 & 24 \end{Vmatrix}$. At point P_3 a sequence of principal minors is:

$|\mathbb{H}_1| = |6| = 6 > 0$, $|\mathbb{H}_2| = \begin{vmatrix} 6 & 0 \\ 0 & 96 \end{vmatrix} = 576 > 0$,

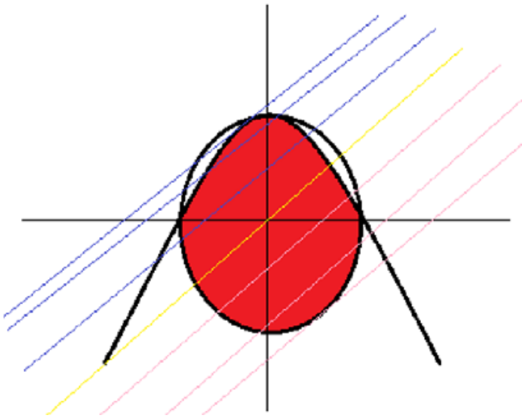
$|\mathbb{H}_3| = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 96 & -24 \\ 0 & -24 & 24 \end{vmatrix} = 10368 > 0$, so we conclude that P_3 is a local minimum point for f.

$\mathbb{H}f(P_4) = \left\| \begin{matrix} -6 & 0 & 0 \\ 0 & 96 & -24 \\ 0 & -24 & 24 \end{matrix} \right\|$. In this case we have a second order principal minor $\mathbb{H}_2 = \left\| \begin{matrix} -6 & 0 \\ 0 & 96 \end{matrix} \right\|$ with a negative determinant equal to $-576 < 0$, and so P_4 is a saddle point.

II M 4) Problem $\begin{cases} \text{Max/min } f(x,y) = y - x \\ \text{u.c.} \begin{cases} x^2 + y^2 \leq 1 \\ y \leq 1 - x^2 \end{cases} \end{cases}$ is equivalent to

$\begin{cases} \text{Max/min } f(x,y) = y - x \\ \text{u.c.} \begin{cases} x^2 + y^2 - 1 \leq 0 \\ x^2 + y - 1 \leq 0 \end{cases} \end{cases}$.

The admissible region is the red-drawed one in the figure:



The objective function is continous and the admissible region is a bounded and closed set, so by Weierstrass Theorem the problem admits absolute maximum and minimum.

The Lagrange function of the problem is:

$\Lambda(x,y,\lambda,\mu) = y - x - \lambda\left(x^2 + y^2 - 1\right) - \mu\left(x^2 + y - 1\right)$ and its gradient is:

$\nabla\Lambda = \left(-1 - 2\lambda x - 2\mu x, 1 - 2\lambda y - \mu, -\left(x^2 + y^2 - 1\right), -\left(x^2 + y - 1\right)\right)$.

KUHN-TUCKER CONDITIONS

First case (*free optimization*): $\begin{cases} \lambda = \mu = 0 \\ -1 = 0 \\ 1 = 0 \\ x^2 + y^2 - 1 \leq 0 \\ x^2 + y - 1 \leq 0 \end{cases}$, the system is impossible.

Second case (*first constraint is active*):

$$\begin{cases} \lambda \neq 0, \mu = 0 \\ -1 - 2\lambda x = 0 \\ 1 - 2\lambda y = 0 \\ x^2 + y^2 - 1 = 0 \\ x^2 + y - 1 \leq 0 \end{cases} \Rightarrow \begin{cases} \mu = 0 \\ x = -1/(2\lambda) \\ y = 1/(2\lambda) \\ 1/(4\lambda^2) + 1/(4\lambda^2) = 1 \\ x^2 + y - 1 \leq 0 \end{cases} \Rightarrow \begin{cases} \mu = 0 \\ x = \mp\sqrt{2}/2 \\ y = \pm\sqrt{2}/2 \\ \lambda = \pm\sqrt{2}/2 \\ 1/2 \pm \sqrt{2}/2 \leq 1 \end{cases};$$

if $\lambda = \sqrt{2}/2$ condition $1/2 + \sqrt{2}/2 \leq 1$ is not satisfied, the system is impossible;

if $\lambda = -\sqrt{2}/2$ condition $1/2 - \sqrt{2}/2 \leq 1$ is satisfied and point $\left(\sqrt{2}/2, -\sqrt{2}/2\right)$ may be a minimum point ($\lambda < 0$).

Third case (*second constraint is active*):

$$\begin{cases} \lambda = 0, \mu \neq 0 \\ -1 - 2\mu x = 0 \\ 1 - \mu = 0 \\ x^2 + y^2 - 1 \leq 0 \\ x^2 + y - 1 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ x = -1/(2\mu) \\ \mu = 1 \\ x^2 + y^2 \leq 1 \\ y = 1 - x^2 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ x = -1/2 \\ \mu = 1 \\ 1/4 + 9/16 \leq 1 \\ y = 3/4 \end{cases},$$

condition $1/4 + 9/16 \leq 1$ is satisfied and point $(-1/2, 3/4)$ may be a maximum point ($\mu > 0$).

Fourth case (*both constraints are active*): $\begin{cases} -1 - 2\lambda x - 2\mu x = 0 \\ 1 - 2\lambda y - \mu = 0 \\ x^2 + y^2 - 1 = 0 \\ x^2 + y - 1 = 0 \end{cases} \Rightarrow$

$$\begin{cases} 2\lambda x + 2\mu x = -1 \\ 2\lambda y + \mu = 1 \\ x^2 + (1 - x^2)^2 - 1 = 0 \\ y = 1 - x^2 \end{cases} \Rightarrow \begin{cases} 2\lambda x + 2\mu x = -1 \\ 2\lambda y + \mu = 1 \\ x^4 - x^2 = 0 \\ y = 1 - x^2 \end{cases} \Rightarrow \begin{cases} 2\lambda x + 2\mu x = -1 \\ 2\lambda y + \mu = 1 \\ x^2(x^2 - 1) = 0 \\ y = 1 - x^2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 2\lambda x + 2\mu x = -1 \\ 2\lambda y + \mu = 1 \\ x = 0 \\ y = 1 \end{cases} \cup \begin{cases} 2\lambda x + 2\mu x = -1 \\ 2\lambda y + \mu = 1 \\ x = 1 \\ y = 0 \end{cases} \cup \begin{cases} 2\lambda x + 2\mu x = -1 \\ 2\lambda y + \mu = 1 \\ x = -1 \\ y = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 0 = -1 \\ \text{impossible} \\ x = 0 \\ y = 1 \end{cases} \cup \begin{cases} \lambda = -3/2 \\ \mu = 1 \\ x = 1 \\ y = 0 \end{cases} \cup \begin{cases} \lambda = -1/2 \\ \mu = 1 \\ x = -1 \\ y = 0 \end{cases}.$$

For both points $(\pm 1, 0)$ the product $\lambda \cdot \mu$ is negative and so the points aren't max or min.

So we have $MAXf = f\left(-\frac{1}{2}, \frac{3}{4}\right) = \frac{5}{4}$; $minf = f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$.

For constraints qualification we consider their Jacobian $J = \left\| \begin{matrix} 2x & 2y \\ 2x & 1 \end{matrix} \right\|$.

In points $\left(-\frac{1}{2}, \frac{3}{4}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ only one constraint is active and trivialy qualified.

In the figure representing the feasible region there are drawn zero level curve (yellow), positive level curves (blue) and negative level curves (pink).